

Solutions

Fall 2011 SM221 Sample Test 4b

1. Use separation of variables to find a non-trivial solution to the PDE:

$$\frac{\partial u}{\partial x} + 3 \frac{\partial u}{\partial y} = 0$$

Note: The trivial solution is the $u = 0$ solution

$$u = XY \Rightarrow \frac{X'Y}{3XY} + \frac{3XY'}{3XY} = 0$$

$$\Rightarrow \frac{X'}{3X} = -\frac{Y'}{Y} = -\lambda \Rightarrow \begin{cases} \frac{X'}{3X} = -\lambda \Rightarrow X' + 3\lambda X = 0 \\ \frac{Y'}{Y} = \lambda \Rightarrow Y' - \lambda Y = 0 \end{cases}$$

$$\therefore X = C_1 e^{-3\lambda x} \quad Y = C_2 e^{\lambda y}$$

$$\Rightarrow u = XY \Rightarrow \boxed{u = k e^{-3\lambda x} e^{\lambda y}}$$

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 $k = C_1 C_2$

or $\boxed{u = k e^{\lambda(y-3x)}}$

2. Use separation of variables to find the general solution to the PDE (use $\lambda > 0$)

$$h \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}, h > 0$$

Note: There are no boundary or initial conditions here, so you can stop at a general solution. You will not be able to find values for λ

$$u = XY \Rightarrow \frac{hX''Y}{hXY} = \frac{XY'}{hXY}$$

$$\Rightarrow \frac{X''}{X} = \frac{Y'}{hY} = -\lambda$$

$$\Rightarrow Y' + h\lambda Y = 0 \Rightarrow \boxed{Y = C_1 e^{-h\lambda y}}$$

$$\Rightarrow X'' + \lambda X = 0 \quad \text{since } -\lambda > 0$$

$$\boxed{X = C_2 \sin(\lambda^{1/2} x) + C_3 \cos(\lambda^{1/2} x)}$$

$$U = XY = \underbrace{C_1 e^{-h\lambda y}}_{a} \underbrace{[C_2 \sin(\lambda^{1/2} x) + C_3 \cos(\lambda^{1/2} x)]}_{\text{combine constants}}$$

$$\boxed{U = e^{-h\lambda y} [a_1 \sin(\lambda^{1/2} x) + a_2 \cos(\lambda^{1/2} x)]}$$

4. Consider the heat equation: $\begin{cases} \frac{\partial^2 u}{\partial x^2} = 3 \frac{\partial u}{\partial t} \\ u_x(0, t) = u_x(5, t) = 0 \\ u(x, 0) = -\frac{1}{2} \cos(\pi x) \end{cases}$

Hint: Part b, the initial condition indicates that you might not have to do a FS expansion.

Hint: Part c, at the middle of the object $x = ???$

- Is the object modeled by the equation insulated? Explain why or why not. \rightarrow yes, derivative boundary conditions
- Solve for temperature function $u(x, t)$.
- Approximate the temperature in the middle of the object at time $t = .1$

$$\frac{x''}{xT} = \frac{3xT'}{XT} \Rightarrow \frac{x''}{x} = \frac{3T'}{T} = -\lambda$$

$$\Rightarrow 3T' + \lambda T = 0 \Rightarrow (3D + \lambda) \cdot 0 = 0 \Rightarrow D = -\frac{1}{3}\lambda \Rightarrow T = a e^{-\frac{1}{3}\lambda t}$$

$$\Rightarrow \frac{x''}{x} = -\lambda \Rightarrow x'' + \lambda x = 0 \quad \begin{cases} \lambda = 0 \Rightarrow x = C_1 + C_2 x \\ \lambda \neq 0 \Rightarrow x = C_3 \sin(\sqrt{\lambda} x) + C_4 \cos(\sqrt{\lambda} x) \end{cases}$$

$$\text{BC } x=0 \quad x' = 0 \Rightarrow x'(0) = C_2 = 0 \Rightarrow x'(5) = C_4 = 0 \quad \checkmark \text{ BC automatically satisfied}$$

$$\therefore x = C_1 \Rightarrow \text{rewrite } x_0 = C_0$$

$$\xrightarrow{\text{BC } x=5} x' = C_1 \sqrt{\lambda} \cos(\sqrt{\lambda} x) - C_2 \sqrt{\lambda} \sin(\sqrt{\lambda} x) = 0$$

$$x'(0) = C_1 \sqrt{\lambda} \cos(0) - C_2 \sqrt{\lambda} \sin(0) = 0 \Rightarrow C_1 = 0$$

$$x'(5) = -C_2 \sqrt{\lambda} \sin(\sqrt{\lambda} \cdot 5) = 0 \Rightarrow 5\sqrt{\lambda} = n\pi \Rightarrow \lambda = \frac{n^2\pi^2}{25}$$

$$\therefore x_n = C_n \cos\left(\frac{n\pi}{5}x\right)$$

$$\lambda = \frac{n^2\pi^2}{25}$$

$$\left\{ \begin{array}{l} T_0 = a e^{-\frac{1}{3}(0)t} = a \Rightarrow T_0 = a_0 \\ x=0 \end{array} \right. \Rightarrow T_0 = a_0 \quad \text{redesignate}$$

$$T_n = a_n e^{-\frac{1}{3} \frac{n^2\pi^2}{25} t} = a_n e^{-\frac{n^2\pi^2}{75} t}$$

$$u = T_0 x_0 + \sum_{n=1}^{\infty} T_n x_n \Rightarrow u = a_0 b_0 + \sum_{n=1}^{\infty} \frac{a_n b_n e^{-\frac{n^2\pi^2}{75} t}}{b_n} \cos\left(\frac{n\pi}{5}x\right)$$

$$\Rightarrow \boxed{u = b_0 + \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{75} t} \cos\left(\frac{n\pi}{5}x\right)}$$

Apply initial condition

$$u(x,0) = b_0 + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{3}x\right) = -\frac{1}{2} \cos(\pi x)$$

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 $n=5$ term

$\therefore b_5 = -\frac{1}{2}$, all other terms $b_n = 0$ & $b_0 = 0$

$$\therefore u(x,t) = -\frac{1}{2} e^{-\frac{\pi^2 t}{75}} \cos(\pi x)$$

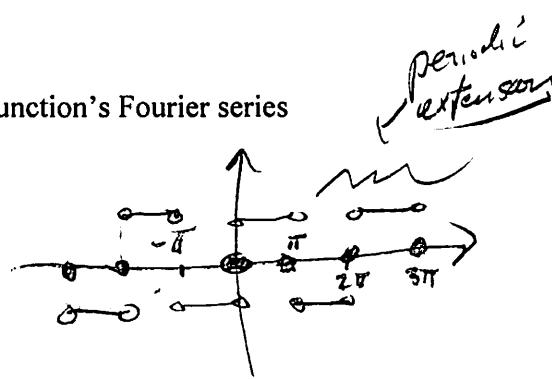
$$\boxed{u(x,t) = -\frac{1}{2} e^{-\frac{\pi^2 t}{75}} \cos(\pi x)}$$

$$u(2.5, 1) = -\frac{1}{2} e^{-\frac{\pi^2 (1)}{75}} \cos(2.5\pi) \neq 0$$

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is way
across here

3. Let: $f(x) = \begin{cases} -1, & -\pi < x \leq 0 \\ 1, & 0 < x < \pi \end{cases}$

- Use the sine or cosine series, as appropriate, to find the function's Fourier series expansion, $FS(x)$.
- Sketch $FS(x)$.
- Find the sum of the first four non-zero terms of $FS\left(\frac{\pi}{2}\right)$.
- Determine the following values exactly:
 - $FS\left(\frac{\pi}{4}\right)$.
 - $FS(0)$
 - $FS(\pi)$



a) Since $f(x)$ is already odd just use the sine series

$$f(x) \sim \sum b_n \sin\left(\frac{n\pi x}{\pi}\right)$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} 1 \sin\left(\frac{n\pi x}{\pi}\right) dx = \frac{2}{n\pi} - \frac{2(-1)^n}{n\pi} = \frac{2}{n\pi} (1 - (-1)^n)$$

$$f(x) \sim \boxed{\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (1 - (-1)^n) \sin(nx)}$$

0 for even terms
2 for odd terms

(b)

$$\frac{2}{\pi} \left[\frac{2}{1} \sin(x) + \frac{2}{3} \sin(3x) + \frac{2}{5} \sin(5x) + \frac{2}{7} \sin(7x) \right]$$

$$\text{or } \boxed{\left[\frac{4}{\pi} \sin(x) + \frac{4}{3\pi} \sin(3x) + \frac{4}{5\pi} \sin(5x) + \frac{4}{7\pi} \sin(7x) \right]}.$$

(c) $\frac{4}{\pi}(1) - \frac{4}{3\pi} + \frac{4}{5\pi} - \frac{4}{7\pi} = \boxed{9216} = \boxed{\frac{3024}{105\pi}}$

(d) $FS\left(\frac{\pi}{2}\right) = 1 \rightarrow FS(0) = 0, FS(\pi) = 0$
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 1-way points

5. Use the function below to find a series that gives the numerical value of $\frac{\pi^2}{3}$:

$$\begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \leq x \leq \pi \end{cases}$$

Hint: First solve the FS for $f(x)$. What should $FS(\pi)$ be equal to? Plug π into $FS(x)$ and see if you can generate a series for $\frac{\pi^2}{8}$

At $x = \pi$ $FS(\pi) = \frac{\pi^2 + 0}{2} = \frac{\pi^2}{2}$

$$\Rightarrow FS(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} 0 dx + \frac{1}{\pi} \int_0^{\pi} x^2 dx = \frac{1}{\pi} \left(\frac{1}{3} \pi^3 \right) = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} x^2 \cos(nx) dx = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} x^2 \sin(nx) dx = -\frac{(-1)^n}{n} + \frac{2(-1)^n}{n^3 \pi} - \frac{2}{n^3 \pi}$$

$$\therefore FS(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left[\frac{2(-1)^n}{n^2} \cos(nx) + \left[\frac{(-1)^{n+1}}{n} + \frac{2(-1)^n}{n^3 \pi} - \frac{2}{n^3 \pi} \right] \sin(nx) \right]$$

$$\therefore FS(\pi) = \frac{\pi^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} (-1)^n + \cancel{\left[x \cancel{\sin(0)} \right]}$$

$$(-1)^n \cdot (-1)^n = 1$$

$$\frac{\pi^2}{2} - \frac{\pi^2}{6} = \boxed{\frac{\pi^2}{3} = \sum_{n=1}^{\infty} \frac{2}{n^2}}$$