

## Fall 2011 SM221 Sample Test 4

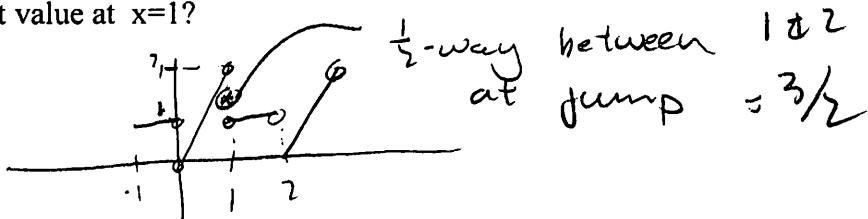
### Multiple Choice

1. The System  $A\vec{x} = \vec{b}$  with  $A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$  has an infinite number of solutions if  $\vec{b} =:$
- a.  $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$       b.  $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$       c.  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$       d.  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
- rrref  $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 6 & -1 & 0 & 1 \\ -1 & -2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{4}{3} \\ 0 & 1 & \frac{4}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- indicated  $\infty$  solutions*

2. Suppose that  $f$  is a function of period 2, and  $f(x) = \begin{cases} 1 & -1 < x < 0 \\ 2x & 0 < x < 1 \end{cases}$ . The Fourier series for  $f$

would converge to what value at  $x=1$ ?

- a. 2  
b. 1  
**c. 1.5**  
d. 0  
e. Undefined



3. The Fourier series for the function defined in the previous problem has coefficient  $a_0$  equal

- to:  
**a. 2**  
b. 1  
c. 1.5  
d. 0  
e. Undefined

$$a_0 = \frac{1}{2} \int_{-1}^0 1 dx + \frac{2}{2} \int_0^1 2x dx = \dots [1+1] = 2$$

4. The Fourier series of a function  $f(x)$  on the interval  $[-\pi, \pi]$  is:

$$1 + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1} = 1 + \left( \frac{4}{\pi} \left[ \sin(x) + \frac{1}{3} \sin(3x) + \frac{1}{5} \sin(5x) + \dots \right] \right)$$

The value of  $\int_{-\pi}^{\pi} f(x) \sin(5x) dx$  is:

- a. 4/5**  
b.  $4/(5\pi)$   
c.  $1/5$   
d. 0

$$P = \pi$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

$$b_5 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(5x) dx$$

$$\Rightarrow b_5 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(5x) dx = \frac{4}{\pi} \frac{1}{5} \Rightarrow \int_{-\pi}^{\pi} f(x) \sin(5x) dx = \frac{4}{5\pi} \pi = \frac{4}{5}$$

Long Answer

1. Use eigenvalues/eigenvectors to find the general solution the initial value problem:

$$\begin{cases} \frac{dx}{dt} = 2x + 2y \\ \frac{dy}{dt} = x + 3y \\ x(0) = 2, \quad y(0) = -1 \end{cases}$$

$$\det \begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = 0 \Rightarrow (6-5\lambda+\lambda^2)-2 = 0 \Rightarrow \lambda^2 - 5\lambda + 4 = 0$$

$$\Rightarrow (\lambda-1)(\lambda-4) = 0 \Rightarrow \lambda = 1, 4$$

$\lambda=1$

$$\begin{bmatrix} 2-1 & 2 \\ 1 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 + 2x_2 = 0 \Rightarrow x_2 = -\frac{1}{2}x_1 \Rightarrow x_1 = 2, x_2 = -1$$

$$\boxed{\vec{v}_{\lambda=1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}}$$

$\lambda=4$

$$\begin{bmatrix} 2-4 & 2 \\ 1 & 3-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -2x_1 + 2x_2 = 0 \Rightarrow x_1 = x_2 \Rightarrow \boxed{\vec{v}_{\lambda=4} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$x = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$$

$$\Rightarrow x = 2c_1 e^t + c_2 e^{4t} \Rightarrow x(0) = 2c_1 + c_2 = 2 \Rightarrow \begin{cases} 2c_1 + c_2 = 2 \\ -c_1 + c_2 = -1 \end{cases}$$

$$y = -c_1 e^t + c_2 e^{4t} \Rightarrow y(0) = -c_1 + c_2 = -1 \Rightarrow \begin{cases} 2c_1 + c_2 = 2 \\ -2c_1 + 2c_2 = -2 \end{cases}$$

$$\Rightarrow 3c_1 = 0 \Rightarrow \boxed{c_1 = 0} \quad \boxed{c_2 = 1}$$

$$\therefore \boxed{x = 2e^t \\ y = -e^t}$$

2. Use Euler's method to approximate the values for  $x(.2)$  and  $y(.2)$ . Let  $\Delta t = .1$ .

$$\begin{aligned}\frac{dx}{dt} &= y + t, & x(0) &= 3 \\ \frac{dy}{dt} &= x + 1, & y(0) &= 4\end{aligned}$$

+	$x$	$y$	$x'$	$y'$	$\Delta t$	$\Delta x$	$\Delta y$
0	3	4	4	4	.1	.4	.4
.1	3.4	4.4	4.5	4.4	.1	.45	.44
.2	3.85	4.84					

3. Use two steps of Euler's method to approximate  $y(1)$ , where:

$$y'' - 3y' + 2y = 1, \quad y(0) = 1, \quad y'(0) = 0$$

$$\begin{aligned}y &= y_1 \\ y' &= y'_1 = y_2 \quad \boxed{y'_1 = y_2} \\ y'' &= y''_1 = y_3 \quad \Rightarrow \quad y_2 - 3y_3 + 2y_1 = 1 \Rightarrow \boxed{y'_2 = 1 + 3y_3 - 2y_1} \\ &\quad \Rightarrow y_1(0)=1, y_2(0)=0\end{aligned}$$

+	$y_1$	$y_2$	$y'_1$	$y'_2$	$\Delta t$	$\Delta y_1$	$\Delta y_2$
0	1	0	0	-1	.5	0	-.5
.5	1	-.5	-.5	-2.5	.5	-.25	-1.25
1.0	-.75	-1.75					

$y(1) = .75$

4. Solve the boundary value problem  $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$  where  $u(0, y) = 5e^{-2y}$ .

$$u = XY$$

$$\Rightarrow \frac{4X'Y + XY'}{4XY} = \frac{3XY'}{4XY} \Rightarrow \frac{X'}{X} + \frac{Y'}{4Y} = \frac{3}{4}$$

$$\therefore \frac{X'}{X} - \frac{3}{4} = -\frac{Y'}{4Y} = k \Rightarrow$$

$$\frac{X'}{X} - \frac{3}{4} = k \Rightarrow X' - \frac{3}{4}X = kX \Rightarrow X' - (\frac{3}{4} + k)X = 0$$

$$\Rightarrow [D - (\frac{3}{4} + k)]X = 0 \Rightarrow D = \frac{3}{4} + k \Rightarrow \boxed{X = ae^{(\frac{3}{4}+k)x}}$$

$$\Rightarrow -\frac{Y'}{4Y} = k \Rightarrow -Y' = 4kY \Rightarrow Y' + 4kY = 0$$

$$\Rightarrow (D + 4k)Y = 0 \Rightarrow D = -4k \Rightarrow Y = be^{-4ky}$$

$$\Rightarrow u = XY = ae^{(\frac{3}{4}+k)x} e^{-4ky}$$

$$u(0, y) = ce^{-4ky} = 5e^{-2y} \Rightarrow \boxed{c=5 \quad k=\frac{1}{2}}$$

$$\therefore u = 5e^{(\frac{3}{4}+\frac{1}{2})x} e^{-2y}$$

$$\Rightarrow \boxed{u = 5e^{1.25x} e^{-2y}} \quad u =$$

5. Consider the function  $f(x)$  defined on the interval  $[-1,1]$  by  $f(x) = \begin{cases} 0 & -1 \leq x < 0 \\ 1-x & 0 \leq x \leq 1 \end{cases}$

- Find the Fourier series of  $f$  on the interval  $[-1,1]$ .
- Sketch the graph of the Fourier series of  $f$  over the interval  $[-3,3]$
- What are the values of the Fourier Series at  $x = -2, 0, 2$ ?

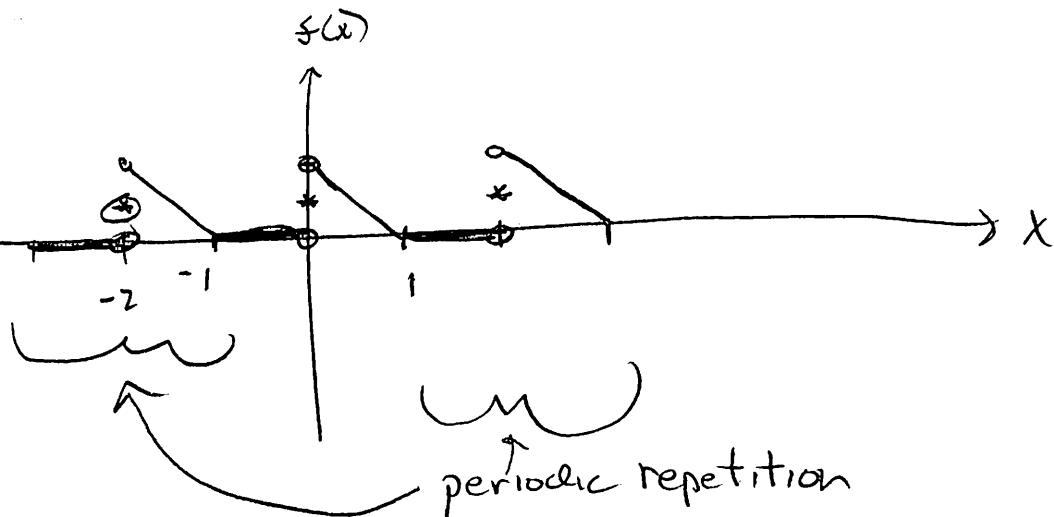
$$a_0 = \frac{1}{\pi} \int_{-1}^0 0 dx + \frac{1}{\pi} \int_0^1 (1-x) dx = \frac{1}{2}$$

$$a_n = \frac{1}{\pi} \int_{-1}^0 0 \cos\left(\frac{n\pi x}{1}\right) dx + \frac{1}{\pi} \int_0^1 (1-x) \cos\left(\frac{n\pi x}{1}\right) dx = \frac{1}{n^2\pi^2} - \frac{(-1)^n}{n^2\pi^2}$$

$$b_n = \frac{1}{\pi} \int_{-1}^0 0 \sin\left(\frac{n\pi x}{1}\right) dx + \frac{1}{\pi} \int_0^1 (1-x) \sin\left(\frac{n\pi x}{1}\right) dx = \frac{1}{n\pi}$$

$\therefore f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( \left[ \frac{1}{n^2\pi^2} (1 - (-1)^n) \right] \cos(n\pi x) + \frac{1}{n\pi} \sin(n\pi x) \right)$

(b)



(c)

$$f(-2) = \frac{1}{2}$$

$$f(0) = \frac{1}{2}$$

$$f(2) = \frac{1}{2}$$

6. Given :  $f(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } 1 < x \leq 2 \end{cases}$

a. Find the Fourier sine series of the function.

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^1 x \sin\left(\frac{n\pi x}{2}\right) dx + \frac{2}{\pi} \int_1^2 1 \sin\left(\frac{n\pi x}{2}\right) dx \\ &= \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi} (-1)^n \end{aligned}$$

$$\therefore f(x) \sim \sum_{n=1}^{\infty} \left[ \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi} (-1)^n \right] \sin\left(\frac{n\pi x}{2}\right)$$

b. Find the Fourier cosine series of the function.

$$a_0 = \frac{2}{\pi} \int_0^1 x dx + \frac{2}{\pi} \int_1^2 1 dx = \frac{3}{2}$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^1 x \cos\left(\frac{n\pi x}{2}\right) dx + \frac{2}{\pi} \int_1^2 \cos\left(\frac{n\pi x}{2}\right) dx = \\ &= \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2} \end{aligned}$$

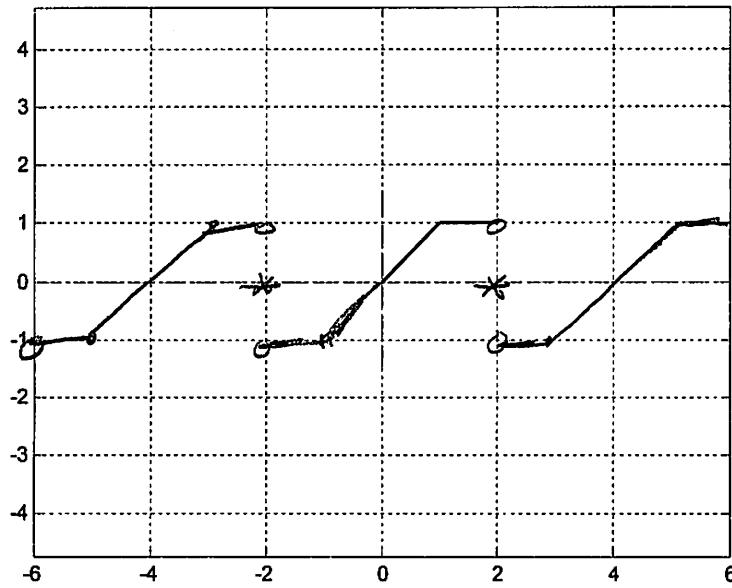
$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \left( \frac{4}{n^2\pi^2} \left[ \cos\left(\frac{n\pi}{2}\right) - 1 \right] \cos\left(\frac{n\pi x}{2}\right) \right)$$



$$a_0/2$$

7. Given :  $f(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } 1 < x \leq 2 \end{cases}$

a. Plot the Fourier sine series of the function.



b. Plot the Fourier cosine series of the function.

