

Score:

Name: SolutionsPeriod (circle one):  1  2  3  4  5  6Team (circle one):  a  b  c  d  e  f**SM212 – Test #4– Spring 2013**

Calculators allowed. YOU MUST SHOW WORK FOR CREDIT.

**BOX YOUR FINAL ANSWER**

1. (15 pts) Find the eigenvalues and eigenvectors for the following matrix:  $\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$

$$\det \begin{bmatrix} 5-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} = (5-\lambda)(4-\lambda) - 6 = 0$$

$$\Rightarrow 20 - 9\lambda + \lambda^2 - 6 = 0 \Rightarrow \lambda^2 - 9\lambda + 14 = 0$$

$$\Rightarrow (\lambda-2)(\lambda-7) = 0 \Rightarrow \boxed{\lambda=2, 7}$$

$$\lambda = 2 \Rightarrow \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow 3x+2y=0 \Rightarrow x = -\frac{2}{3}y$$

$$\therefore \boxed{\vec{v}_{\lambda=2} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}}$$

$$\lambda = 7 \Rightarrow \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -2x+2y=0 \Rightarrow x=y$$

$$\boxed{\vec{v}_{\lambda=7} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

No marks on this table	
ST (25pts)	
1 (15 pts)	
2 (15 pts)	
3 (15 pts)	
4 (15 pts)	
5 (15 pts)	

2. (10 pts) Using your results from problem 1, solve the following system of differential equations. Find  $x(2)$ ,  $y(2)$ .

$$\begin{aligned}\frac{dx}{dt} &= 5x + 2y, & x(0) &= -2 \\ \frac{dy}{dt} &= 3x + 4y, & y(0) &= 3\end{aligned}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t}$$

$$\begin{aligned} \Rightarrow x &= 2C_1 e^{2t} + C_2 e^{7t} \\ y &= -3C_1 e^{2t} + C_2 e^{7t} \end{aligned} \quad \begin{array}{l} \text{Initial} \\ \xrightarrow{\text{conditions}} \end{array} \quad \begin{aligned} x(0) &= 2C_1 + C_2 = -2 \\ y(0) &= -3C_1 + C_2 = 3 \end{aligned}$$

$$\begin{array}{l} 5C_1 = -5 \Rightarrow C_1 = -1 \\ C_2 = 0 \end{array}$$

$x = -2e^{2t}$
$y = 3e^{2t}$

$$\begin{aligned} x(2) &= -2e^4 \Rightarrow x(2) = -2,984 \\ y(2) &= 3e^4 \Rightarrow y(2) = 4,475 \end{aligned} \quad \boxed{\begin{array}{l} x(2) = -2,984 \\ y(2) = 4,475 \end{array}}$$

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3. (15 pts) Use Euler's method to approximate the values for  $x(.2)$  and  $y(.2)$ . Let  $\Delta t = .1$ . Why are these answers different from those in Problem 2.

$$\frac{dx}{dt} = 5x + 2y, \quad x(0) = -2$$

$$\frac{dy}{dt} = 3x + 4y, \quad y(0) = 3$$

$t$	$x$	$y$	$x'$	$y'$	$\Delta t$	$\Delta x$	$\Delta y$
0	-2	3	-4	6	.1	-.4	.6
.1	-2.4	3.6	-4.8	7.2	.1	-.48	.72

$$.2 \quad -2.88 \quad 4.32$$

$$x(.2) = -2.88$$

$$y(.2) = 4.32$$

Euler's Method is an approximation.  
Problem 2 was solved analytically  
& will produce a more correct  
answer.

4. (15 pts) Given  $f(x) = (1-x)x$  for  $0 \leq x \leq 1$

- Find a general expression for the Fourier sine series of  $f(x)$ .
- Write out the first 3 non-zero terms of the series.
- Carefully sketch the series on the interval  $[-3, 3]$ . (Use the graph below)

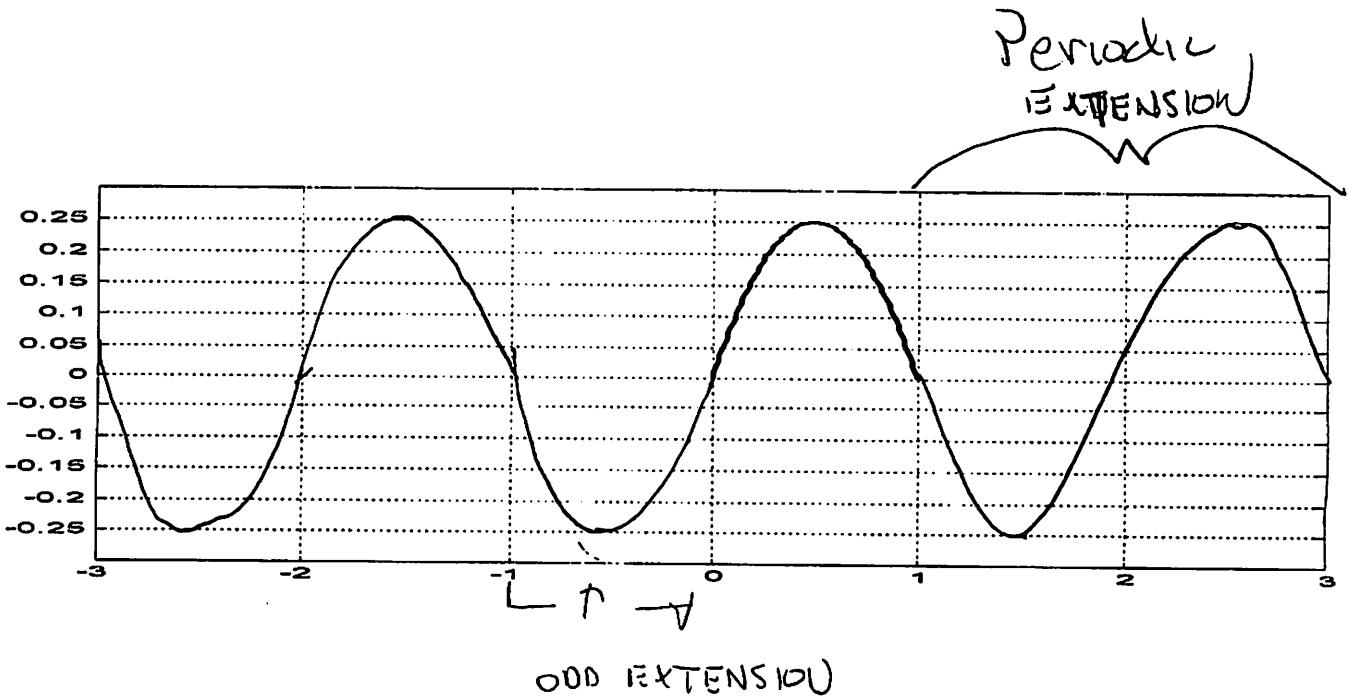
Recall:  $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$  where  $b_n = \frac{2}{p} \int_0^p f(x) \sin\left(\frac{n\pi x}{p}\right) dx$

$$b_n = \frac{2}{1} \int_0^1 (1-x)x \sin(n\pi x) dx = -\frac{4(-1)^n}{n^3 \pi^3} + \frac{4}{n^3 \pi^3}$$

$$\therefore b_n = \frac{4}{n^3 \pi^3} [1 - (-1)^n] = \begin{cases} 0 & n \rightarrow \text{even} \\ \frac{8}{n^3 \pi^3} & n \rightarrow \text{odd} \end{cases}$$

④  $f(x) \sim \left[ \sum_{n=1}^{\infty} \frac{4}{n^3 \pi^3} [1 - (-1)^n] \sin(n\pi x) \right]$

$$f(x) = \frac{8}{\pi^3} \sin(\pi x) + \frac{16}{27\pi^3} \sin(3\pi x) + \frac{8}{125\pi^3} \sin(5\pi x) \dots$$



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5. (15 pts) Write out the first three non-zero terms for the solution to the boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ with boundary conditions } u(0, t) = 0 \text{ and } u(1, t) = 0 \text{ and initial condition}$$

$u(x, 0) = (1 - x)x$ . Assume that  $\lambda > 0$  are the only eigenvalues that will produce a non-trivial solution.

$$u = XT \Rightarrow \frac{XT'}{XT} = \frac{X''T}{XT} \Rightarrow \frac{T'}{T} = \frac{X''}{X} = -\lambda \Rightarrow \begin{cases} T' + \lambda T = 0 \\ X'' + \lambda X = 0 \end{cases}$$

$$\Rightarrow \begin{cases} T = a_1 e^{-\lambda t} \\ X = c_1 \sin(\lambda^{\frac{1}{2}} x) + c_2 \cos(\lambda^{\frac{1}{2}} x) \end{cases} \text{ since } \lambda > 0$$

$$\text{B.C.s } X(0) = C_1(0) + C_2(1) = 0 \Rightarrow C_2 = 0$$

$$X(1) = C_1 \sin(\lambda^{\frac{1}{2}}) = 0 \Rightarrow \lambda^{\frac{1}{2}} = n\pi \Rightarrow \lambda = n^2\pi^2$$

$$\therefore X_n = c_n \sin(n\pi x)$$

$$T_n = a_n e^{-n^2\pi^2 t}$$

$$\Rightarrow u_n = T_n X_n = a_n c_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

$$\Rightarrow u = \sum_{n=1}^{\infty} u_n \Rightarrow \boxed{u = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin(n\pi x)}$$

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \Rightarrow b_n = \int_0^1 (1-x)x \sin(n\pi x) dx$$

$\hookrightarrow b_n$  from prob #4

$$\Rightarrow u(x, t) = \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3} [1 - (-1)^n] e^{-n^2\pi^2 t} \sin(n\pi x)$$

$$u(x, t) \approx \frac{8}{\pi^3} e^{-\pi^2 t} \sin(\pi x) + \frac{8}{27\pi^3} e^{-9\pi^2 t} \sin(3\pi x) + \frac{8}{125\pi^3} e^{-25\pi^2 t} \sin(5\pi x)$$