

Score:

100

Name: SolutionsSection (circle one): 1 2 3 4 5 6
Team (circle one): a b c d e f**SM212 – Test #3– Spring 13**Calculators allowed. Box/circle your final answer.
YOU MUST SHOW ALL WORK FOR FULL CREDIT.

1. Given $f(t) = \begin{cases} t^2 & 1 \leq t < 3 \\ t & t \geq 3 \end{cases}$

- a. (5 pts) Write $f(t)$ as a step function.

$$f(t) = t^2 u(t-1) + (t - t^2) u(t-3)$$

- b. (5 pts) What is $\mathcal{L}\{f(t)\}$? Simplify your answer if possible.

$$\begin{aligned} & \mathcal{L}\{(t+1)^2\} e^{-s} + \mathcal{L}\{(t+3) - (t+3)^2\} e^{-3s} \\ &= \mathcal{L}\{t^2 + 2t + 1\} e^{-s} + \mathcal{L}\{t+3 - (t^2 + 6t + 9)\} e^{-3s} \\ &= \mathcal{L}\{t^2 + 2t + 1\} e^{-s} + \mathcal{L}\{t - e^2 - 5e^{-3s}\} e^{-3s} \end{aligned}$$

$$= \left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \right) e^{-s} + \left(-\frac{2}{s^3} - \frac{5}{s^2} - \frac{6}{s} \right) e^{-3s}$$

No marks on this table	
ST (20 pts)	
1 (10 pts)	
2 (15 pts)	
3 (10 pts)	
4 (10 pts)	
5 (10 pts)	
6 (10 pts)	
7 (15 pts)	
cumm.	

2. Inverse Laplace Transforms:

a. (5 pts) 1st Translation Theorem: Determine $\mathcal{L}^{-1}\left\{\frac{s+6}{s^2+4s+20}\right\}$

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s+6}{(s^2+4s+4)+16}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+6}{(s+2)^2+4^2}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2+4^2}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{(s+2)^2+4^2}\right\} \\ &= e^{-2t} \cos(\frac{4}{2}t) + e^{-2t} \sin(\frac{4}{2}t) \end{aligned}$$

b. (5 pts) Unit Step Functions: Determine $\mathcal{L}^{-1}\left\{e^{-2\pi s} \frac{s}{s^2+4}\right\}$

$$\cos(2(t-2\pi)) u(t-2\pi)$$

$$= \cos(2t - 4\pi) u(t-2\pi)$$

$$= \boxed{\cos(2t) u(t-2\pi)}$$

c. (5 pts) Convolution: Determine $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+1)^2}\right\}$

$$\mathcal{L}^{-1}\left\{\left(\frac{s}{s^2+1}\right)\left(\frac{s}{s^2+1}\right)\right\} = \cos(t) \otimes \cos(t)$$

$$= \int_0^t \cos(\tau) \cos(t-\tau) d\tau = \boxed{\frac{1}{2} t \cos(t) + \frac{1}{2} \sin(t)}$$

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3. Solve:

a. (5 pts) Use *Laplace transforms* to solve: $y' - 3y = \delta(t)$, $y(0) = 0$.

$$sY(s) - y(0) - 3Y(s) = 1$$

$$\Rightarrow (s-3)Y(s) = 1 \Rightarrow Y(s) = \frac{1}{s-3}$$

$$\Rightarrow \boxed{y(t) = e^{3t}}$$

↑
Green's function $g(t)$

b. (5 pts) Use answer from part a and convolution to solve: $y' - 3y = e^{3t}$, $y(0) = 0$

↑
 $f(t)$

$$y(t) = g(t) * f(t)$$

$$= e^{3t} * e^{3t} \Rightarrow \int_0^t e^{3x} e^{3(t-x)} dx$$

I can do
this integral
by hand,

$$= \int_0^t e^{3x} e^{3t-3x} dx = \int_0^t e^{3t} dx$$

$$= e^{3t} x \Big|_{x=0}^{x=t} \Rightarrow \boxed{(te^{3t})}$$

4. (10 pts) Use *Laplace transforms* to solve: $y' - 3y = e^{3t}$, $y(0) = 1$.

$$\cancel{SY(s) - y(0)} - 3Y(s) = \frac{1}{s-3}$$

$$\Rightarrow (S-3)Y(s) = \frac{1}{s-3} + 1$$

$$\Rightarrow Y(s) = \frac{1}{(s-3)^2} + \frac{1}{s-3}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= e^{3t} \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + e^{3t}$$

$$\Rightarrow \boxed{te^{3t} + e^{3t}}$$

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5. (10 pts) Use **Laplace transforms** to solve: $y'' + 16y = U(t - \pi)$, $y(0) = 1$, $y'(0) = -2$.

$$\begin{aligned} s^2 Y(s) - s y(0) - y'(0) + 16Y(s) &= \frac{e^{-\pi s}}{s} \\ \Rightarrow (s^2 + 16) Y(s) &= \frac{e^{-\pi s}}{s} + s - 2 \\ \Rightarrow Y(s) &= e^{-\pi s} \left(\frac{1}{s(s^2 + 16)} \right) + \frac{s}{s^2 + 16} - \frac{2}{s^2 + 16} \\ &\quad \text{EXPNND} \\ \Rightarrow Y(s) &= \frac{1}{16} \frac{e^{-\pi s}}{s} + \frac{1}{16} e^{-\pi s} \left(\frac{s}{s^2 + 16} \right) + \frac{s}{s^2 + 16} - \frac{1}{2} \frac{4}{s^2 + 16} \\ \Rightarrow y(t) &= \frac{1}{16} u(t - \pi) + \frac{1}{16} \cos(4(t - \pi)) u(t - \pi) \\ &\quad + \cos(4t) - \frac{1}{2} \sin(4t) \end{aligned}$$

$$\Rightarrow \boxed{y(t) = \frac{1}{16} (\cos(4t) + \cos(4(t - \pi)) + \sin(4(t - \pi)))}$$

6. (10 pts) Solve the systems of equations

$$\begin{cases} \frac{dx}{dt} = -5x - 4y - e^t, & x(0) = 1 \\ \frac{dy}{dt} = -2x - 3y + t, & y(0) = 0 \end{cases}$$

$$\left\{ \begin{array}{l} \frac{dx}{dt} + 5x + 4y = -e^t \\ 2x + \frac{dy}{dt} + 3y = t \end{array} \right\}$$

$$\begin{cases} sX(s) - x(0)^{(1)} + 5X(s) + 4Y(s) = -\frac{1}{s-1} \\ 2X(s) + sY(s) - y(0)^{(1)} + 3Y(s) = \frac{1}{s^2} \end{cases}$$

$$\begin{bmatrix} s+5 & 4 & 1 - \frac{1}{s-1} \\ 2 & s+3 & \frac{1}{s^2} \end{bmatrix}$$

\Rightarrow RREF / EXPAND

$$\Rightarrow \begin{cases} X(s) = \frac{449}{588} \frac{1}{s+7} - \frac{1}{6} \frac{1}{s+1} - \frac{1}{4} \frac{1}{(s-1)} + \frac{32}{49} \frac{1}{s} - \frac{4}{7} \frac{1}{s^2} \\ Y(s) = \frac{449}{1176} \frac{1}{s+7} + \frac{1}{6} \frac{1}{s+1} + \frac{1}{8} \frac{1}{(s-1)} - \frac{33}{49} \frac{1}{s} + \frac{5}{7} \frac{1}{s^2} \end{cases}$$

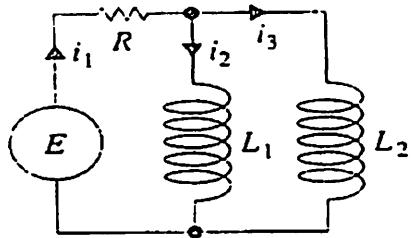
$$\boxed{\begin{aligned} X(t) &= \frac{449}{588} e^{-7t} - \frac{1}{6} e^{-t} - \frac{1}{4} e^t + \frac{32}{49} - \frac{4}{7} t \\ Y(t) &= \frac{449}{1176} e^{-7t} + \frac{1}{6} e^{-t} + \frac{1}{8} e^t - \frac{33}{49} + \frac{5}{7} t \end{aligned}}$$

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7. (15 pts) Solve the system depicted on the right.

Let $R = 20$ ohms, $L_1 = 5$ henrys, $L_2 = 4$ henrys, $E = 100$ volts, $i_2(0) = i_3(0) = 0$ amps.

Determine, $i_1(t)$, $i_2(t)$, and $i_3(t)$.



$$\left\{ \begin{array}{l} Ri_1 + L_1 i_2' = E \\ Ri_1 + L_2 i_3' = E \\ i_1 = i_2 + i_3 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} 20i_1 + 5i_2' = 100 \\ 20i_1 + 4i_3' = 100 \\ i_1 - i_2 - i_3 = 0 \end{array} \right. \quad \left\{ \begin{array}{l} 20i_1 + 5i_2' = 100 \\ 20i_1 + 4i_3' = 100 \\ i_1 - i_2 - i_3 = 0 \end{array} \right. \quad \left\{ \begin{array}{l} 20i_1 + 5i_2' = 100 \\ 20i_1 + 4i_3' = 100 \\ i_1 - i_2 - i_3 = 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} 20I_1(s) + 5sI_2(s) - I_2(0) = \frac{100}{s} \\ 20I_1(s) + 4sI_3(s) - I_3(0) = \frac{100}{s} \\ I_1(s) - I_2(s) - I_3(s) = 0 \end{array} \right.$$

$$\left[\begin{array}{ccc|c} 20 & 5s & 0 & \frac{100}{s} \\ 20 & 0 & 4s & \frac{100}{s} \\ 1 & -1 & -1 & 0 \end{array} \right] \xrightarrow{\text{RREF}} \text{EXPAND}$$

$$I_1(s) = \frac{5}{s} - \frac{5}{s+4}$$

$$I_2(s) = \frac{20}{9} \cdot \frac{1}{s} = \frac{20}{9} \cdot \frac{1}{s+4} \Rightarrow$$

$$I_3(s) = \frac{25}{9} \cdot \frac{1}{s} - \frac{25}{9} \cdot \frac{1}{s+4}$$

$$\boxed{\begin{aligned} i_1(t) &= 5 - 5e^{-4t} \\ i_2(t) &= \frac{20}{9} \cdot \frac{20}{9} e^{-4t} \\ i_3(t) &= \frac{25}{9} - \frac{25}{9} e^{-4t} \end{aligned}}$$