

1. Find the following:

a.  $L\{\sin(\beta t)\} = \frac{\beta}{s^2 + \beta^2}$

b.  $L\{e^{\alpha t} \sin(\beta t)\} = \frac{\beta}{(s-\alpha)^2 + \beta^2}$

c.  $L\{2te^{\alpha t} \sin(\beta t)\}$

$$2 L\{t e^{\alpha t} \sin(\beta t)\}$$

$$= 2(-1)^n \frac{d}{ds} \left( \frac{\beta}{(s-\alpha)^2 + \beta^2} \right)$$

$$= -2 \frac{((s-\alpha)^2 + \beta^2)(0) - \beta(2)(s-\alpha)}{((s-\alpha)^2 + \beta^2)^2}$$

$$\frac{4\beta(s-\alpha)}{((s-\alpha)^2 + \beta^2)^2}$$

Name: \_\_\_\_\_

2. Find  $L\{g(t)\}$  where  $g(t) = \begin{cases} 1 & \text{for } 0 < t < 1 \\ 1 & \text{for } 1 < t < 2 \\ 2 & \text{for } t > 2 \end{cases}$

$$\mathcal{L} \left\{ 1 - u(t-1) + t u(t-1) - t u(t-2) + 2 u(t-2) \right\}$$

$$\frac{1}{s} - \frac{e^{-s}}{s} + e^{-s} \mathcal{L}\{t+1\} - e^{-2s} \mathcal{L}\{t+2\} + 2 \frac{e^{-2s}}{s}$$

$$= \frac{1}{s} - \frac{e^{-s}}{s} + e^{-s} \left( \frac{1}{s^2} + \frac{1}{s} \right) - e^{-2s} \left\{ \frac{1}{s^2} + \frac{2}{s} \right\} + 2 \frac{e^{-2s}}{s}$$

$$= \frac{1}{s} + e^{-s} \left[ \frac{1}{s^2} + \frac{1}{s} \right] + e^{-2s} \left[ -\frac{1}{s^2} - \frac{2}{s} + \frac{2}{s} \right]$$

$$= \frac{1}{s} + e^{-s} \left( \frac{1}{s^2} \right) - e^{-2s} \left( \frac{1}{s^2} \right)$$

3. Given  $f(t) = \begin{cases} 0 & 0 < t < 5 \\ 10 & t > 5 \end{cases}$ , answer the following.

a. Use the definition of the Laplace Transform to determine  $L\{f(t)\}$ .

$$\int_5^{\infty} 10e^{-st} dt = \frac{10}{s} e^{-st} \Big|_5^{\infty}$$
$$= -\frac{10}{s} (e^{-\infty} - e^{-5s}) = \frac{10e^{-5s}}{s}$$

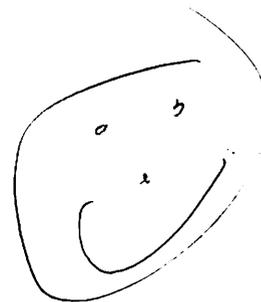
b. Express  $f(t)$  as a step function.

$$f(t) = 10u(t-5)$$

c. Find the Laplace transform of the step function in part 'b' using table values. How does your answer compare to the answer in part 'a'.

$$\boxed{\frac{10e^{-5s}}{s}}$$

SAME



Name: \_\_\_\_\_

4. Find the following:

$$\begin{aligned}
 \text{a. } L^{-1}\left\{\frac{s+2}{s^2+4s+20}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+2}{(s^2+4s+4)+16}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2+16}\right\} \\
 &= \boxed{e^{-2t} \cos(4t)}
 \end{aligned}$$

$$\text{b. } L^{-1}\left\{\frac{2}{(s+5)^3}\right\} = e^{-5t} \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \boxed{t^2 e^{-5t}}$$

Step

$$\text{c. } L^{-1}\left\{e^{-2s} \left(\frac{1}{s^2+2}\right)\right\} = \sin(\sqrt{2}) \cdot \boxed{\frac{1}{\sqrt{2}} \sin(\sqrt{2}(t-2)) u(t-2)}$$

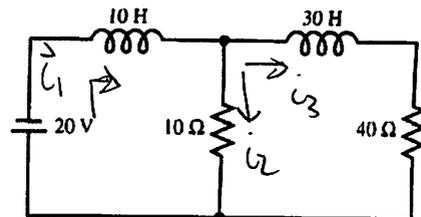
$$u(t-2) \frac{1}{\sqrt{2}} \mathcal{L}^{-1}\left\{\frac{1 \times \sqrt{2}}{s^2+2}\right\} = \boxed{u(t-2) \frac{1}{\sqrt{2}} \sin(\sqrt{2}(t-2))}$$

$$\text{d. } L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} \cdot \frac{s}{s^2+1}\right\}$$

$$= \sin(t) * \cos(t)$$

$$= \int_0^{\infty} \sin(t\tau) \cos(t-\tau) d\tau = \frac{1}{2} t \sin(t)$$

5. Solve for the  $I(t)$  in the circuit depicted on the right. Initially there is not current in the system. What is the current in the center branch as  $t \rightarrow \infty$ .



$$10 \dot{i}_1 + 10 \dot{i}_2 = 20$$

$$-10 \dot{i}_2 + 30 \dot{i}_3 + 40 \dot{i}_3 = 0$$

$$i_1 = i_2 + i_3$$

$$i_1(0) = 0$$

$$i_2(0) = 0$$

$$i_3(0) = 0$$

$$\left\{ \begin{array}{l} 10 \dot{i}_2 + 10 \dot{i}_3 + 10 i_2 = 20 \\ -10 \dot{i}_2 + 30 \dot{i}_3 + 40 i_3 = 0 \end{array} \right.$$

$$s(10I_2(s) - i_2(0)) + 10(I_3(s) - i_3(0)) + 10I_2(s) = \frac{20}{s}$$

$$-10I_2(s) + 30sI_3(s) - i_2(0) + 40I_3(s) = 0$$

$$(10s + 10)I_2(s) + 10sI_3(s) = \frac{20}{s}$$

$$-10I_2(s) + 30sI_3(s) + 40I_3(s) = 0$$

$$\begin{bmatrix} 10s+10 & 10s & 20/s \\ -10 & 30s+40 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \text{EXPAND}$$

$$I_2(s) = -\frac{9}{2} \frac{1}{3s+2} - \frac{1}{2} \frac{1}{s+2} + \frac{2}{s} = -\frac{9}{2} \frac{1}{3(s+2/3)} + \frac{1}{2} \frac{1}{s+2} + \frac{2}{s}$$

$$I_3(s) = -\frac{9}{4} \frac{1}{3(s+2/3)} + \frac{1}{4} \frac{1}{s+2} + \frac{1}{2s}$$

$$\Rightarrow \left. \begin{array}{l} i_2(t) = -\frac{3}{2} e^{-2/3t} - \frac{1}{2} e^{-2t} + 2 \\ i_3(t) = -\frac{3}{4} e^{-2/3t} + \frac{1}{4} e^{-2t} + \frac{1}{2} \end{array} \right\} \text{center branch current}$$

$$i_2(\infty) = 2$$

Name: \_\_\_\_\_

6. (10 pts) Let  $A = \begin{bmatrix} -1 & 3 \\ -4 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$ . Find the following:

a.  $2A + 3B$

$$\begin{bmatrix} -2 & 6 \\ -8 & 4 \end{bmatrix} + \begin{bmatrix} -6 & 12 \\ 3 & -6 \end{bmatrix} = \boxed{\begin{bmatrix} -8 & 18 \\ -5 & -2 \end{bmatrix}}$$

b.  $AB$

$$\begin{bmatrix} -1 & 3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2+3 & -4-6 \\ 8+2 & -16-4 \end{bmatrix} = \boxed{\begin{bmatrix} 5 & -10 \\ 10 & -20 \end{bmatrix}}$$

c.  $BA$

$$\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 2-16 & -6+8 \\ -1+8 & 3-4 \end{bmatrix} = \boxed{\begin{bmatrix} -14 & 2 \\ 7 & -1 \end{bmatrix}}$$

d.  $CA$  if  $C = \text{inv}(A)$

$$CA = \text{inv}(A)A = A^{-1}A = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

e.  $\text{inv}(B)$

$$\det(B) = (-2)(-2) - (4)(1) = 0 \Rightarrow \text{since } \det(B) = 0 \text{ no inverse!!}$$

f.  $\text{inv}(A)$

Cramer's Rule!

$$\det(A) = -2+12=10 \Rightarrow A^{-1} = \frac{1}{10} \begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} .2 & -.3 \\ .4 & -.1 \end{bmatrix}}$$

7. Consider the system of differential equations (where  $x$  and  $y$  are function of  $t$ ). Solve the IVP system using Laplace transforms.

$$\begin{aligned}x' &= 2x - 3y, & x(0) &= 5 \\y' &= x - 2y, & y(0) &= 3\end{aligned}$$

$$\begin{aligned}5X(s) - \overset{5}{x(0)} &= 2X(s) - 3Y(s) \\5Y(s) - \overset{3}{y(0)} &= X(s) - 2Y(s)\end{aligned} \Rightarrow \begin{aligned}(s-2)X(s) + 3Y(s) &= 5 \\-X(s) + (s+2)Y(s) &= 3\end{aligned}$$

$$\Rightarrow \begin{bmatrix} s-2 & 3 & 5 \\ -1 & s+2 & 3 \end{bmatrix} \xrightarrow[\text{EXPAND}]{\text{RREF}}$$

$$X(s) = \frac{2}{s+1} + \frac{3}{s-1} \Rightarrow$$

$$Y(s) = \frac{2}{s+1} + \frac{1}{s-1}$$

$$\boxed{\begin{aligned}x(t) &= 2e^{-t} + 3e^t \\y(t) &= 2e^{-t} + e^t\end{aligned}}$$

8. Use Laplace transforms to solve the differential equation  $y' + y = \delta(t-1)$  where  $y(0) = 1$ .

$$sY(s) - \overset{1}{y(0)} + Y(s) = e^{-s}$$

$$\Rightarrow (s+1)Y(s) = e^{-s} + 1 \Rightarrow Y(s) = e^{-s} \left( \frac{1}{s+1} \right) + \frac{1}{s+1}$$

$$\Rightarrow \boxed{y(t) = e^{-(t-1)} u(t-1) + e^{-t}}$$

$$\boxed{y(t) = \begin{cases} e^{-t} & 0 \leq t < 1 \\ e^{-t} + e^{-(t-1)} & t \geq 1 \end{cases}}$$

Name: \_\_\_\_\_

9. Use Laplace transforms to solve the differential equation  $y'' + y = \cos(t)$  where  $y(0) = 1$  and  $y'(0) = 1$ .

$$s^2 Y(s) - \cancel{s y(0)} - \cancel{y'(0)} + Y(s) = \frac{s}{s^2+1}$$

$$\Rightarrow (s^2+1)Y(s) = \frac{s}{s^2+1} + s + 1 \Rightarrow Y(s) = \frac{s}{(s^2+1)^2} + \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$\Rightarrow Y(s) = \left(\frac{s}{s^2+1}\right)\left(\frac{1}{s^2+1}\right) + \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$y(t) = \underset{\substack{\uparrow \\ \text{convolution}}}{\cos(t) * \sin(t)} + \cos(t) + \sin(t)$$

$$\Rightarrow y(t) = \int_0^t \cos(\tau) \sin(t-\tau) d\tau + \cos(t) + \sin(t)$$

$$\Rightarrow y(t) = \frac{1}{2} t \sin(t) + \cos(t) + \sin(t)$$

10. Solve the DE  $y' - 2y = 3U(t-4)$  where  $y(0) = 1$ .

$$sY(s) - \cancel{y(0)} - 2Y(s) = \frac{3e^{-4s}}{s}$$

$$(s-2)Y(s) = 3e^{-4s} \left(\frac{1}{s}\right) + 1$$

$$\begin{aligned} \Rightarrow Y(s) &= 3e^{-4s} \left(\frac{1}{s(s-2)}\right) + \frac{1}{s-2} \\ &= 3e^{-4s} \left(\frac{1}{2} \frac{1}{s-2} - \frac{1}{2} \frac{1}{s}\right) + \frac{1}{s-2} \end{aligned}$$

$$\Rightarrow y(t) = \frac{3}{2} (e^{2(t-4)} - 1) U(t-4) + e^{2t}$$

11. Consider the system of D.E.'s  $\begin{cases} x' = -4x - y + 2 \\ y' = -7x + 2y - t \end{cases}$  with  $x(0) = -1$  and  $y(0) = 1$ . Solve the system using Laplace transforms (Don't let messy fractions scare you!!).

$$\begin{aligned} sX(s) - \overset{-1}{x(0)} &= -4X(s) - Y(s) + \frac{2}{s} \\ sY(s) - \underset{1}{y(0)} &= -7X(s) + 2Y(s) - \frac{1}{s^2} \end{aligned}$$

$$\begin{aligned} (s+4)X(s) + Y(s) &= \frac{2}{s} - 1 \\ 7X(s) + (s-2)Y(s) &= -\frac{1}{s^2} + 1 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} s+4 & 1 & \frac{2}{s}-1 \\ 7 & s-2 & -\frac{1}{s^2}+1 \end{bmatrix}$$

$$X(s) = \frac{-221}{200} \frac{1}{s+5} - \frac{11}{72} \frac{1}{s-3} + \frac{58}{225} \frac{1}{s} - \frac{1}{15} \frac{1}{s^2}$$

$$Y(s) = \frac{-221}{200} \frac{1}{s+5} + \frac{77}{72} \frac{1}{s-3} + \frac{233}{225} \frac{1}{s} + \frac{4}{15} \frac{1}{s^2}$$

$$\Rightarrow \begin{cases} x(t) = \frac{-221}{200} e^{-5t} - \frac{11}{72} e^{3t} + \frac{58}{225} - \frac{1}{15} t^2 \\ y(t) = \frac{-221}{200} e^{-5t} + \frac{77}{72} e^{3t} + \frac{233}{225} + \frac{4}{15} t \end{cases}$$

$$x(0) = \frac{-221}{200} - \frac{11}{72} + \frac{58}{225} = -1 \checkmark \checkmark$$

$$y(0) = \frac{-221}{200} + \frac{77}{72} + \frac{233}{225} = 1 \checkmark \checkmark$$

CHECK

Name: \_\_\_\_\_

12. Find Green's function for the DE  $x'' + 4x$ .

$$q'' + 4q = \delta(t) \Rightarrow q(0) = 0, q'(0) = 0$$
$$(s^2 + 4)G(s) = 1 \Rightarrow G(s) = \frac{1}{s^2 + 4}$$

$$\Rightarrow \boxed{q(t) = \frac{1}{2} \sin(2t)}$$

a. Use Green's function to solve for  $x'' + 4x = e^{-2t}, x(0) = 0, x'(0) = 0$ .

$$x(t) = e^{-2t} * \frac{1}{2} \sin(2t) = \frac{1}{2} \int_0^t e^{-2\tau} \sin(2(t-\tau)) d\tau$$

$$\boxed{x(t) = -\frac{1}{8} \cos(2t) + \frac{1}{8} \sin(2t) + \frac{1}{8} e^{-2t}}$$

b. Use Green's function to solve for  $x'' + 4x = \sin(2t), x(0) = 0, x'(0) = 0$ .

$$\sin(2t) * \frac{1}{2} \sin(2t) = \frac{1}{2} \int_0^t \sin(2\tau) * \sin(2(t-\tau)) d\tau$$

$$= \boxed{\frac{1}{8} \sin(2t) - \frac{1}{4} t \cos(2t)}$$