

Score:

Name: _____

Section (circle one): 1 2 3 4 5 6
Team (circle one): a b c d e f

SM212 – Test #2– Spring 2013

Calculators allowed. Box/circle your final answer.
YOU MUST SHOW ALL WORK FOR FULL CREDIT.

1. (a) Given $f(t) = \begin{cases} 1 & 0 \leq t \leq 4 \\ 0 & t > 4 \end{cases}$, use the definition of the Laplace transform to find $\mathcal{L}\{f(t)\}$.

$$\int_0^4 e^{-st} dt + \cancel{\int_4^\infty e^{-st} dt} \xrightarrow{10} = -\frac{1}{s} e^{-st} \Big|_0^4$$

$$= -\frac{1}{s} e^{-4s} - \left(-\frac{1}{s} e^0 \right) = \boxed{\frac{1}{s} - \frac{1}{s} e^{-4s}}$$

- (b) For the function above, use the first translation theorem to find $\mathcal{L}\{e^{2t}f(t)\}$.

rep 's' w/ $s-2$

$$\boxed{\frac{1}{s-2} - \frac{1}{s-2} e^{-4(s-2)}}$$

No marks on this table	
ST (10 pts)	
1 (10 pts)	
2 (15 pts)	
3 (10 pts)	
4 (10 pts)	
5 (15 pts)	
6 (15 pts)	
7 (15 pts)	
cumm.	

2. (15 pts) (a) (Hint: 1st Translation Theorem) Determine $\mathcal{L}^{-1}\left\{\frac{3}{(s+2)^4}\right\}$

$$e^{-2t} \mathcal{L}^{-1}\left\{\frac{3}{s^4}\right\} = \frac{3}{3!} e^{-2t} \mathcal{L}^{-1}\left\{\frac{1 \times 3!}{s^4}\right\}$$

$$= \boxed{\frac{1}{2} e^{-2t} t^3}$$

(b) Determine $\mathcal{L}^{-1}\left\{\frac{1}{s^2 - 8s + 15}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s-5)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2} \frac{1}{s-5} - \frac{1}{2} \frac{1}{s-3}\right\}$$

$$= \boxed{\frac{1}{2} e^{5t} - \frac{1}{2} e^{3t}}$$

(c) Determine $\mathcal{L}^{-1}\left\{\frac{s+5}{s^2 + 8s + 17}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s+5}{(s^2 + 8s + 16) + 17 - 16}\right\} = \mathcal{L}^{-1}\left\{\frac{s+5}{(s+4)^2 + 1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s+4)}{(s+4)^2 + 1} + \frac{1}{(s+4)^2 + 1}\right\}$$

$$= \boxed{e^{-4t} \cos(t) + e^{-4t} \sin(t)}$$

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3. (10 pts) Use **Laplace transforms** to solve: $y'' + 16y = 0$, $y(0) = -2$, $y'(0) = 8$
 (Hint: $\mathcal{L}\{0\} = 0$)

$$s^2 Y(s) - sy(0) - y'(0) + 16Y(s) = 0$$

$$\Rightarrow (s^2 + 16)Y(s) - s(-2) - 8 = 0 \Rightarrow (s^2 + 16)Y(s) = -2s + 8$$

$$\Rightarrow Y(s) = \frac{-2s}{s^2 + 16} + \frac{8}{s^2 + 16}$$

$$\Rightarrow y(t) = -2 \mathcal{F}^{-1} \left\{ \frac{s}{s^2 + 16} \right\} + 2 \mathcal{F}^{-1} \left\{ \frac{4}{s^2 + 16} \right\}$$

$$\Rightarrow \boxed{y(t) = -2\cos(4t) + 2\sin(4t)}$$

4. (10 pts) Solve: $y'' + 9y = 0$, $y\left(\frac{\pi}{3}\right) = 3$, $y'\left(\frac{\pi}{3}\right) = 12$. Hint: you cannot use Laplace transforms since initial values are not defined at $t = 0$.

$$(D^2 + 9)y = 0 \Rightarrow D = \pm 3i \Rightarrow y = C_1 \cos(3t) + C_2 \sin(3t)$$

$$\Rightarrow y\left(\frac{\pi}{3}\right) = C_1 \cos\cancel{\left(\pi\right)}^{\cancel{-1}} - C_2 \sin\cancel{\left(\pi\right)}^{\cancel{+6}} = 3 \Rightarrow \boxed{C_1 = -3}$$

$$\therefore y = -3\cos(3t) + C_2 \sin(3t)$$

$$\Rightarrow y' = 9\sin(3t) + 3C_2 \cos(3t)$$

$$\Rightarrow y'\left(\frac{\pi}{3}\right) = 9\sin\cancel{\left(\pi\right)}^{\cancel{+1}} + 3C_2 \cos\cancel{\left(\pi\right)}^{\cancel{-1}} = 12 \Rightarrow \boxed{C_2 = -4}$$

$$\therefore \boxed{y = -3\cos(3t) - 4\sin(3t)}$$

5. (15 pts) Use *undetermined coefficients* to solve: $y'' - 16y = 16\cos(4t)$, $y(0) = 0$, $y'(0) = 1$.

$$\textcircled{1} \quad (D^2 - 16)y_H = 0 \Rightarrow (D+4)(D-4)y_H = 0 \Rightarrow D = -4, 4$$

$$\Rightarrow y_H = C_1 e^{-4t} + C_2 e^{4t}$$

$$\textcircled{2} \quad y_p = A \cos(4t) + B \sin(4t) \quad \textcircled{3} \quad \text{OK}$$

$$\textcircled{4} \quad y_p' = -4A \sin(4t) + 4B \cos(4t)$$

$$y_p'' = -16A \cos(4t) - 16B \sin(4t)$$

$$\Rightarrow y_p'' - 16y_p = -32A \cos(4t) - 32B \sin(4t) = 16\cos(4t)$$

$$\Rightarrow B=0, \quad A = -\frac{1}{2}$$

$$\textcircled{5} \quad y = C_1 e^{-4t} + C_2 e^{4t} - \frac{1}{2} \cos(4t)$$

$$y(0) = C_1 + C_2 - \frac{1}{2} = 0 \Rightarrow C_1 + C_2 = \frac{1}{2}$$

$$\Rightarrow y' = -4C_1 e^{-4t} + 4C_2 e^{4t} + 2 \sin(4t)$$

$$\Rightarrow y'(0) = -4C_1 + 4C_2 + 0 = 1 \Rightarrow$$

$$\Rightarrow C_1 + C_2 = \frac{1}{2} \times 4 \Rightarrow 4C_1 + 4C_2 = 2$$

$$-4C_1 + 4C_2 = 1$$

$$\underline{-4C_1 + 4C_2 = 1}$$

$$8C_2 = 3 \Rightarrow C_2 = \frac{3}{8}$$

$$\boxed{y = \frac{1}{8} e^{-4t} + \frac{3}{8} e^{4t} - \frac{1}{2} \cos(4t)}$$

$$C_1 = \frac{1}{8}$$

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6. (15 pts) Use **Laplace transforms** to solve: $y'' - 16y = 16 \cos(4t)$, $y(0) = 0$, $y'(0) = 1$.

$$s^2 Y(s) + s\cancel{y(0)} - \cancel{y'(0)} - 16 Y(s) = \frac{16s}{s^2 + 16}$$

$$\Rightarrow (s^2 - 16) Y(s) = \frac{16s}{s^2 + 16} + 1$$

$$\Rightarrow Y(s) = \frac{16s}{(s^2 + 16)(s^2 - 16)} + \frac{1}{s^2 - 16}$$

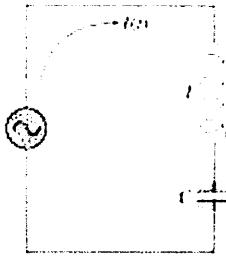
$$\Rightarrow Y(s) = -\frac{1}{2} \frac{s}{s^2 + 16} + \frac{1}{8} \frac{1}{s+4} + \frac{3}{8} \frac{1}{s-4}$$

$$\Rightarrow \boxed{y(t) = -\frac{1}{2} \cos(4t) + \frac{1}{8} e^{-4t} + \frac{3}{8} e^{4t}}$$

7. (15 pts) The differential equation that models a series LC circuit is given by $L \frac{d^2q}{dt^2} + \frac{1}{C} q = E(t)$. If $L=2$ h, $C=.125$ f, and $E(t)=18$ V.

Note: q is charge, $\frac{dq}{dt}$ is current.

- Find $q(t)$ if initially there is no charge and current.
- If $E(t)=\sin(\gamma t)$ V, for what value of γ will resonance occur?



(a) $2q'' + 8q = 18 \Rightarrow q'' + 4q = 9 \quad q(0)=0, q'(0)$

$$s^2 Q(s) - s q(0) - q'(0) + 4Q(s) = \frac{9}{s}$$

$$\Rightarrow (s^2 + 4)Q(s) = \frac{9}{s} \Rightarrow Q(s) = \frac{9}{s(s^2 + 4)}$$

$$\Rightarrow Q = \frac{9}{4s} - \frac{9}{4} \frac{s}{s^2 + 4} \Rightarrow$$

$$q(t) = \frac{9}{4} - \frac{9}{4} \cos(2t)$$

(b) $\boxed{\gamma = 2}$