

Score:

Name: Solutions

Section (circle one): 1 2 3 4 5 6
Team (circle one): a b c d e f

SM212 – Sample Test 2 – Fall 2011

1. A possible form for a particular solution of $y'' + 2y = \cos(\sqrt{2}t)$ is

- a. $A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t)$
- b. $e^{\sqrt{2}t} A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t)$
- c. $t(A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t))$
- d. None of them

$$(D^2 + 2)y_p = 0 \Rightarrow D = \pm i\sqrt{2}$$
$$\Rightarrow Y_H = C_1 \sin(\sqrt{2}t) + C_2 \cos(\sqrt{2}t)$$
$$\Rightarrow Y_p = A + \underset{t \text{ fix}}{\sin(\sqrt{2}t)} + Bt \cos(\sqrt{2}t)$$

2. $\mathcal{L}\{e^{t+3}\} =$

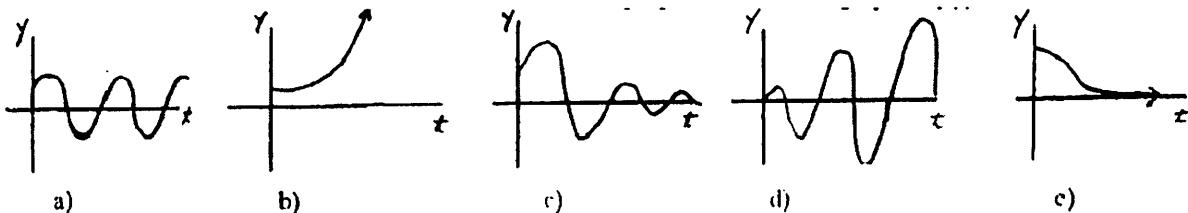
- a. $\frac{1}{s+2}$
- b. $\frac{1}{s-3}$
- c. $e^{3t}U(t+3)$
- d. $\frac{e^3}{s-1}$

$$\mathcal{L}\{e^t e^3\} = e^3 \mathcal{L}\{e^t\} = \frac{e^3}{s-1}$$

3. $\mathcal{L}^{-1}\left\{\frac{s}{s^2-6s+10}\right\} =$

- a. $e^{3t}\cos(t)$
- b. $e^{-3t}\cos(t)$
- c. $e^{-3t}(\cos(t) + 3\sin(t))$
- d. $e^{3t}(\cos(t) + 3\sin(t))$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2-6s+10)+10}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s-3)^2+1}\right\}$$
$$= \mathcal{L}\left\{\frac{\cancel{s-3}}{(s-3)^2+1} + \frac{3}{(s-3)^2+1}\right\}$$
$$= e^{3t}\cos(t) + 3e^{3t}\sin(t)$$



4. If there is not external force ($F(t)=0$) and no damping ($b=0$), then could look like: \Rightarrow undamped free motion

- a' b c d e

5. If $F(t) = 10\cos(\omega t)$ and $b=0$ results in resonance, $y(t)$ could look like:

- a b c d e

increasing amplitude!

6. If $F(t)=0$ and $b>0$ results in under damped motion, $y(t)$ could look like:

- a b c' d e

a... under damped
or critical damped
or over damped

7. The function $f(t) = \begin{cases} 1, & 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$ has the following Laplace transform:

- a. $\int_0^\infty e^{-st} dt$
- b. $\int_0^\infty te^{-st} dt$
- c. $\int_1^2 tdt$
- d. $\int_1^2 te^{-st} dt$
- e. $\int_1^2 e^{-st} dt$

$$\cancel{\int_0^1 e^{-st} (0) dt} + \int_1^2 e^{-st} (t) dt + \cancel{\int_2^\infty e^{-st} (0) dt}$$

$$\int_1^2 e^{-st} dt$$

8. The Laplace transform of $f(t)$ in the problem above is:

- a. $\frac{e^{-1s} - e^{-2s}}{s}$
- b. $\frac{e^{2s} - e^s}{s}$
- c. $\frac{e^{-2s} - 1}{s}$
- d. $u(t-1)$
- e. $\frac{1}{s}$

$$-\frac{1}{s} e^{-st} \Big|_1^2 = -\frac{1}{s} (e^{-2s} - e^{-s})$$

$$= \frac{e^{-s} - e^{-2s}}{s}$$

9. The Laplace transform of $\mathcal{L}\{e^{2t} \sin(3t)\}$ is:

- a. $\frac{s}{(s-2)^2 + 9}$
- b. $\frac{3}{(s-2)^2 + 9}$
- c. $\frac{e^{(s-2)}}{(s-2)^2 + 9}$
- d. $\frac{e^{2s}}{(s-2)^2 + 9}$
- e. $\frac{2}{(s-3)^2 + 4}$

$$\frac{1}{s^2 + 9} \xrightarrow{\text{shift}} \frac{3}{(s-2)^2 + 9}$$

10. The inverse Laplace transform $\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 4s + 8}\right\}$ equals:

- a. $\cos(2t)$
- b. $e^{-2t} \cos(2t)$
- c. $e^{-2t} \cos(2t)$
- d. $\sin(2t)$
- e. $e^{-2t} \cos(2t) - 2e^{-2t} \sin(2t)$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2 + 4s + 4) + 4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2 + 4}\right\} = \frac{1}{2} \cos(2t) - \frac{1}{2} \sin(2t)$$

11. A mass weighing 4 pounds stretches a spring 6 inches. Take $g = 32 \text{ ft/s}^2$ to be the acceleration due to gravity at the earth's surface. The displacement $x(t)$ of the mass from equilibrium satisfies the differential equation:

- a. $x'' + 64x = 0$
- b. $x'' + 2x = 0$
- c. $x'' + (1/6)x = 0$
- d. $4x'' + (2/3)x = 0$

$$F = kx \Rightarrow 4 = k(\frac{1}{2} \cdot 6) \Rightarrow k = 8$$

$$F = mg \Rightarrow m = \frac{F}{g} = \frac{4}{32} = \frac{1}{8}$$

$$\Rightarrow \frac{1}{4}x'' + 8x = 0 \Rightarrow x'' + 32x = 0$$

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12. An undamped mass-spring system with external force is governed by the differential equation $x'' + 9x = \sin(yt)$. Resonance occurs when $\gamma =$

- a. 9
- b. 3
- c. 1
- d. ω^2

$$\Rightarrow D^2 + 9 = 0 \Rightarrow D = \pm\sqrt{3}$$

resonance freq

13. Match the non-homogeneous expression on the left with the guess on the right that you would use when using the method of undetermined coefficients.

e t^3

a. $At^2 e^{\beta t} + Bte^{\beta t} + Ce^{\beta t}$

g $t \sin(\beta t)$

b. $A \sin(\beta t) + B \cos(\beta t)$

h $t^2 e^{\beta t}$

c. none

b $\cos(\beta t)$

d. $Ae^{\beta t} + Be^{-\beta t}$

f $e^{\alpha t} \sin(\beta t)$

e. $At^3 + Bt^2 + Ct + D$

d $e^{\beta t} + e^{-\beta t}$

f. $Ae^{\alpha t} \sin(\beta t) + Be^{\alpha t} \cos(\beta t)$

c $\alpha \ln(\beta t)$

g. $At \sin(\beta t) + Bt \cos(\beta t) + C \sin(\beta t) + D \cos(\beta t)$

14. Solve: $D(D-1)(D+2)^2 y = e^{-2t} + t^2 + e^{2t}$. Identify the homogeneous and particular solutions. Do not solve for the coefficients!

$$D=0, 1, -2, -2 \Rightarrow y_h = C_1 + C_2 e^t + C_3 e^{-2t} + C_4 t e^{-2t}$$

\star \star \star fix

$$\Rightarrow y_p = Ae^{-2t} + Bt^2 + Ct + D + Ee^{2t}$$

$$\Rightarrow \boxed{y_p = At^2 e^{-2t} + Bt^3 + Ct^2 + Dt + E e^{2t}}$$

15. (20 points) The behavior of an LCR circuit is governed by the differential equation:

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = E(t)$$

- L = inductance
- R = resistance
- C = capacitance
- q = charge
- E(t) = voltage source

If the circuit contains a source $E(t) = \sin(t)$, a 1 Henry inductor, 1 farad capacitor, and no resistance:

- Determine the charge $q(t)$ if the system is initially uncharged ($q(0) = 0$) and has no current ($q'(0) = 0$).
- What physically happens to the system at $t \rightarrow \infty$?

$$q'' + q = \sin(t) \Rightarrow (D^2 + 1) q_H = 0 \Rightarrow D = \pm i$$

$$\Rightarrow q_H = C_1 \sin(t) + C_2 \cos(t)$$

$$q_P = A \sin(t) + B \cos(t)$$

$$q'_P = A \sin(t) + A \cos(t) + B \cos(t) - B \sin(t)$$

$$q''_P = 2A \cos(t) - A \sin(t) - 2B \sin(t) - B \cos(t)$$

$$\Rightarrow q''_P + q_P = 2A \cos(t) - 2B \sin(t) = \sin(t) \Rightarrow B = -\frac{1}{2}$$

$$\therefore q_P = -\frac{1}{2} \cos(t)$$

$$\Rightarrow q = q_H + q_P = C_1 \sin(t) + \cancel{C_2 \cos(t)} - \frac{1}{2} \cos(t)$$

$$q(0) = C_1(0) + C_2(1) - \cancel{\frac{1}{2}(0)} = 0 \Rightarrow C_2 = 0$$

$$q' = C_1 \cos(t) - \frac{1}{2} \cos(t) + \cancel{\frac{1}{2} t \sin(t)} =$$

$$q'(0) = C_1 - \frac{1}{2} - 0 = 0 \Rightarrow C_1 = \frac{1}{2}$$

$$\boxed{q = \frac{1}{2} \sin(t) - \frac{1}{2} t \cos(t)}$$

$$\textcircled{b} \quad q(\infty) \rightarrow \infty \Rightarrow \text{"Wettklang"}$$

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16. Use **undetermined coefficients** to solve the initial value problem: $y'' - 3y' + 2y = 4e^t$,
 $y(0) = 2, y'(0) = 1$.

$$\textcircled{1} \quad (D^2 - 3D + 2)y_H = 0 \Rightarrow (D - 1)(D - 2)y_H = 0 \Rightarrow D=1, D=2$$

$$y_H = C_1 e^t + C_2 e^{2t}$$

$$\textcircled{2} \quad y_p = Ae^t \xrightarrow{\text{F.I.}} \textcircled{3} \quad y_p = Ate^t$$

$$y'_p = Ae^t + Ate^t$$

$$y''_p = 2Ae^t + Ate^t$$

$$\textcircled{4} \quad y''_p - 3y'_p + 2y_p = 2Ae^t + Ate^t - 3Ae^t - 3Ate^t + 2Ate^t = 4e^t$$

$$\Rightarrow -At^2 = 4e^t \Rightarrow A = -4 \Rightarrow \boxed{y_p = -4te^t}$$

$$\textcircled{5} \quad y = C_1 e^t + C_2 e^{2t} - 4te^t \Rightarrow y(0) = C_1 + C_2 = 2 \Rightarrow C_1 + C_2 = 2 \Rightarrow \boxed{C_1 = 3, C_2 = -1}$$

$$y' = C_1 e^t + 2C_2 e^{2t} - 4e^t - 4te^t \quad y'(0) = C_1 + 2C_2 - 4 = 1 \Rightarrow C_1 + 2C_2 = 5$$

$$\Rightarrow \boxed{y = -4te^t - e^t + 3e^{2t}}$$

17. Use **Laplace Transforms** to solve the initial value problem:

$$y'' - 3y' + 2y = 4e^t, \quad y(0) = 2, \quad y'(0) = 1.$$

$$s^2 Y(s) - s y(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{4}{s-1}$$

$$\Rightarrow s^2 Y(s) - 2s - 1 - 3sY(s) + 6 + 2Y(s) = \frac{4}{s-1}$$

$$\Rightarrow (s^2 - 3s + 2)Y(s) = \frac{4}{s-1} + 2s - 5$$

$$\Rightarrow Y(s) = \left(\frac{4}{s-1} + 2s - 5 \right) \frac{1}{(s-1)(s+2)} = -\frac{1}{s-1} - \frac{4}{(s-1)^2} + \frac{3}{s+2}$$

EXPANDED $\frac{1}{(s-1)(s+2)}$

$$y(t) = -e^t + 3e^{2t} - 4 \int \left\{ \frac{1}{(s-1)^2} \right\}$$

$$= -4e^t \int \left\{ \frac{1}{s^2} \right\}$$

$$\boxed{y(t) = -e^t + 3e^{2t} - 4te^t} \checkmark$$

18. Use **Laplace transforms** to find the solution to:

$$y''' - 3y'' + 2y' = 4x - e^{-x}, y(0) = y'(0) = y''(0) = 0.$$

$$\begin{aligned} s^3 Y(s) - \cancel{\frac{2}{s} y(0)} - \cancel{s y'(0)} - \cancel{y''(0)} - 3 \left[s^2 Y(s) - \cancel{s y(0)} - \cancel{s y'(0)} \right] + 2 \left[Y(s) - y(0) \right] \\ = \frac{4}{s^2} - \frac{1}{s+1} \\ (s^3 - 3s^2 + 2s) Y(s) = \frac{4}{s^2} - \frac{1}{s+1} \Rightarrow Y(s) = \left(\frac{4}{s^2} - \frac{1}{s+1} \right) \left(\frac{1}{s^3 - 3s^2 + 2s} \right) \\ \Rightarrow Y(s) = \frac{1}{6} \frac{1}{s+1} - \frac{3}{2} \frac{1}{s-1} + \frac{1}{3} \frac{1}{s^2} + \frac{3}{s} + \frac{3}{s^2} + \frac{2}{s^3} \\ \Rightarrow \boxed{y(t) = \frac{1}{6} e^{-t} - \frac{3}{2} e^t + \frac{1}{3} t^2 + 3 + 3t + t^2} \end{aligned}$$

19. Solve the DE $y'' + y = 0$, $y\left(\frac{\pi}{2}\right) = 2$, $y'\left(\frac{\pi}{2}\right) = -1$

$$(D^2 + 1)y = 0 \Rightarrow y(t) = C_1 \sin(t) + C_2 \cos(t)$$

$$y\left(\frac{\pi}{2}\right) = C_1(1) + C_2(0) = 2 \Rightarrow C_2 = 2$$

$$y = 2 \sin(t) + C_2 \cos(t)$$

$$\Rightarrow y' = 2 \cos(t) - C_2 \sin(t)$$

$$y'\left(\frac{\pi}{2}\right) = 2(0) - C_2(1) = -1 \Rightarrow C_2 = 1$$

$$\Rightarrow \boxed{y = 2 \sin(t) + \cos(t)}$$

$\omega = 1$

$$A = (2^2 + 1^2)^{1/2} = \sqrt{5}, \quad \phi = \tan^{-1}\left(\frac{1}{2}\right) = .464$$

$$\therefore \boxed{y = \sqrt{5} \sin(t + .464)}$$

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20. The displacement of a weight in an undamped mass-spring system subject to an external force $F(t) = 8\sin(2t)$ is governed by the differential equation: $x'' + 4x = 8\sin(2t)$.

a. Assume the initial conditions $x(0)=0, x'(0)=1$, use undetermined coefficients to solve the initial value problem.

$$\textcircled{1} \quad (0^2 + 4)x_H = 0 \Rightarrow D = \pm 2i \Rightarrow x_H = C_1 \sin(2t) + C_2 \cos(2t)$$

$$\textcircled{2} \Rightarrow x_P = A \underset{\textcircled{3}}{\sin(2t)} + B \underset{\textcircled{3}}{\cos(2t)} \quad \textcircled{3} \underset{F_{1x}}{F_{1x}} \underset{y_P}{y_P} //$$

$$\textcircled{4} \quad x'_P = A \sin(2t) + 2At \cos(2t) + B \cos(2t) - 2Bt \sin(2t)$$

$$\begin{aligned} x''_P &= 2A \cos(2t) + 2A \cos(2t) - 4At \sin(2t) \\ &\quad - 2B \sin(2t) - 2B \sin(2t) - 4Bt \cos(2t) \\ &= 4A \cos(2t) - 4At \sin(2t) - 4B \sin(2t) - 4Bt \cos(2t) \end{aligned}$$

$$x''_P + 4x_P = 4A \cos(2t) - 4B \sin(2t)$$

$$\Rightarrow 4A = 0 \Rightarrow \boxed{A=0} \quad -4B = 8 \Rightarrow \boxed{B=-2}$$

$$\boxed{x_P = -2t \cos(2t)}$$

$$\textcircled{5} \quad X = C_1 \sin(2t) + \cancel{C_2 \cos(2t)} - 2t \cos(2t)$$

$$\Rightarrow X(0) = C_2 - 0 = 0 \Rightarrow \boxed{C_2 = 0}$$

$$\Rightarrow X = C_1 \sin(2t) - 2t \cos(2t)$$

$$\Rightarrow X' = 2C_1 \cos(2t) - 2 \cos(2t) + 4t \sin(2t)$$

$$\Rightarrow X'(0) = 2C_1 - 2 + 0 = 1 \Rightarrow 2C_1 = 3 \Rightarrow \boxed{C_1 = \frac{3}{2}}$$

$$\therefore \boxed{X = \frac{3}{2} \sin(2t) - 2t \cos(2t)}$$

21. A 1-kg mass is attached to a spring hanging from the ceiling, thereby causing the spring to stretch .392 m upon coming to rest at equilibrium. At time $t = 0$, the mass is displaced .125 m below the equilibrium position and released. At this same instant, an external force $F(t) = 10 \cos t$ N is applied to the system.

- If the damping constant for the system is 8 N-sec/m, determine the equation of motion for the mass.
- What is the resonance frequency of the system?

\uparrow

$x=0$

$m=1$

$b=8$

$F=kx \Rightarrow (1)(9.8) = k(0.392) \Rightarrow k = 25$

$mx'' + bx' + kx = F(t)$

$\Rightarrow x'' + 8x' + 25x = 10 \cos t, \quad x(0) = -0.125, \quad x'(0) = 0$

$\mathcal{L}\{x\} = s^2 X(s) - s \cancel{x(0)} - \cancel{x'(0)} + 8(sX(s) - \cancel{x(0)}) + 25X(s) = \frac{10s}{s^2 + 1}$

$= s^2 X(s) - \frac{1}{8}s + 8sX(s) - 1 + 25X(s) = \frac{10s}{s^2 + 1}$

$= (s^2 + 8s + 25)X(s) = \frac{10s}{s^2 + 1} + \frac{1}{8}s + 1$

$= X(s) = \left(\frac{10s}{s^2 + 1} + \frac{1}{8}s + 1\right) \left(\frac{1}{s^2 + 8s + 25}\right)$

complete square

$s^2 + 8s + 25 = (s+4)^2 + 9$

$= -\frac{1}{4} \left(\frac{(s+4)}{(s+4)^2 + 9} - \frac{4}{3} \frac{3}{(s+4)^2 + 9}\right) - \frac{17}{8 \cdot 3} \left(\frac{1 \cdot 3}{(s+4)^2 + 9}\right) + \frac{3}{8} \frac{s}{s^2 + 1} + \frac{1}{8} \frac{1}{s^2 + 1}$

$X(t) = -\frac{1}{4} e^{-4t} \cos(3t) + \left(\frac{1}{3} e^{-4t} 64 \operatorname{arctan}(3t) - \frac{17}{24} e^{-4t} \sin(3t)\right) + \frac{3}{8} \cos(t) + \frac{1}{8} \sin(t)$

$$X(t) = -\frac{1}{4} e^{-4t} \cos(3t) - \frac{3}{8} e^{-4t} \sin(3t) + \frac{3}{8} \cos(t) + \frac{1}{8} \sin(t)$$

#21) By Undetermined Coefficients

$$\textcircled{1} \quad x_H'' + 8x_H' + 25x_H = 0 \Rightarrow -\frac{-8 \pm \sqrt{64-100}}{2} = -\frac{-8 \pm \sqrt{-36}}{2} = \frac{-8 \pm 6i}{2} = -4 \pm 3i$$

$$\Rightarrow x_H = C_1 e^{-4t} \sin(3t) + C_2 e^{-4t} \cos(3t)$$

$$\textcircled{2} \quad x_p = A \cos(t) + B \sin(t)$$

\hookrightarrow $\textcircled{3}$ no redundancies

$$\textcircled{4} \quad x_p' = -A \sin(t) + B \cos(t)$$

$$x_p'' = -A \cos(t) - B \sin(t)$$

$$\Rightarrow (-A + 8B + 25A) \cos(t) + (-B - 8A + 25B) \sin(t) = 10 \cos(t)$$

$$\Rightarrow 24A + 8B = 10$$

$$-8A + 24B = 0 \quad \times 3$$

$$24A + 8B = 10$$

$$-24A + 72B = 0$$

$$80B = 10 \Rightarrow B = \frac{1}{8}$$

$$24B = 8A \Rightarrow A = 3B \Rightarrow A = \frac{3}{8}$$

$$\therefore x_p = \frac{3}{8} \cos(t) + \frac{1}{8} \sin(t)$$

$$\Rightarrow x = C_1 e^{-4t} \sin(3t) + C_2 e^{-4t} \cos(3t) + \frac{3}{8} \cos(t) + \frac{1}{8} \sin(t)$$

$$x(0) = C_1(1)(0) + C_2(1)(1) + \frac{3}{8} = \frac{1}{8} \Rightarrow C_2 = -\frac{1}{4}$$

$$\Rightarrow x = C_1 e^{-4t} \sin(3t) - \frac{1}{4} e^{-4t} \cos(3t) + \frac{3}{8} \cos(t) + \frac{1}{8} \sin(t)$$

$$x' = -4C_1 e^{-4t} \sin(3t) + 3C_1 e^{-4t} \cos(3t)$$

$$+ e^{-4t} \cos(3t) + \frac{3}{4} e^{-4t} \sin(3t) - \frac{3}{8} \sin(t) + \frac{1}{8} \cos(t)$$

$$\Rightarrow x'(0) = 3C_1 + 1 + \frac{1}{8} = 0 \Rightarrow 3C_1 = -\frac{9}{8} \Rightarrow C_1 = -\frac{3}{8}$$

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22. An RLC series circuit has a constant voltage source given by $E(t)=60V$, a resistor of $R=14$ ohms, and inductor of $L=2 H$, and a capacitor of $C=1/20 F$. If the initial current is zero, and the initial charge on the capacitor is zero:

- Set up an IVP for the charge $q(t)$ on the capacitor for $t>0$.
- Use Laplace transforms to solve the IVP in part(a) and determine the transient charge and the steady state charge on the capacitor.
- Determine the current of $t>0$.

$$(a) Lq'' + Rq' + \frac{1}{C}q = 60 \Rightarrow 2q'' + 14q' + 2q = 60, q(0)=0, q'(0)=0$$

$$\Rightarrow q'' + 7q' + 10q = 30 \Rightarrow 2\{ \quad * \quad \} \Rightarrow$$

$$s^2Q(s) - s q(0) - q'(0) + 7(sQ(s) - q(0)) + 10Q(s) = \frac{30}{5}$$

$$(s^2 + 7s + 10)Q(s) = \frac{30}{5} \Rightarrow Q(s) = \frac{30}{(s)s^2 + 7s + 10}$$

$$\Rightarrow Q(s) = 2\left(\frac{1}{s+5}\right) - 5\left(\frac{1}{s+2}\right) + 3\left(\frac{1}{s}\right)$$

$$\Rightarrow q(t) = 2e^{-5t} - 5e^{-2t} + 3$$

$$(b) q(\infty) = 0 - 0 + 3 = 3 \text{ coulombs}$$

$$-10e^{-5t} + 10e^{-2t} = i(t)$$

$$i(t) = q'(t)$$