

Score:

100

Name: Solutions

Section (circle one): 1 2 3 4 5 6

Team (circle one): a b c d e f

## SM212 – Test 1 – Spring 2013

1. (15 pts) Theory:

- a. Classify the DE:  $x^4 \frac{d^3y}{dx^3} + x^3 \frac{d^2y}{dx^2} + x^2 \frac{dy}{dx} + xy^2 = \sin(x)$ .

Order:	3rd	Linear/Non-Linear:	non-linear
Dependent Variable:	y	Homogeneity:	non-homog.
Independent Variable:	x	Ordinary/Partial:	ordinary

- b. Given:  $(x - 3) \frac{d^2y}{dx^2} + \frac{1}{x-\pi} \frac{dy}{dx} + \frac{1}{x^2-1} y = 0, y(0) = 0, y'(0) = 1$ . Identify the intervals of  $x$  for which a unique solution is guaranteed.

$$x \neq 3, x \neq \pi, x \neq 1, x \neq -1$$

or

$$(-\infty, -1) \cup (-1, 1) \cup (1, 3) \cup (3, \pi) \cup (\pi, \infty)$$

- c. The solutions for the DE  $y'' + k^2y = 0$  are  $y = \sin(kx)$  and  $y = \cos(kx)$ . Show that this solution set is linearly independent (Hint: Wronskian).

$$W = \begin{vmatrix} \sin(kx) & \cos(kx) \\ k\cos(kx) & -k\sin(kx) \end{vmatrix} =$$

$$\begin{aligned} & -k\sin^2(kx) - k\cos^2(kx) \\ &= -k(\sin^2(kx) + \cos^2(kx)) \\ &= -k(1) = -k \neq 0 \end{aligned}$$

No marks on this table	
MC (20pts)	
1 (15pts)	
2 (10 pts)	
3 (15 pts)	
4 (10 pts)	
5 (10 pts)	
6 (10 pts)	
7 (10 pts)	
cumm.	

therefore linearly independent!

2. (10 pts) Show that  $y = c_1 + c_2x + c_3e^{2x} + c_4e^{-2x}$  is a solution to the differential equation  $\frac{d^4y}{dx^4} - 4\frac{d^2y}{dx^2} = 0$ , in two different ways:

a. Show the solution is correct by substituting it into the DE:

$$y' = c_2 + 2c_3e^{2x} - 2c_4e^{-2x}$$

$$y'' = 4c_3e^{2x} + 4c_4e^{-2x}$$

$$y''' = 8c_3e^{2x} - 8c_4e^{-2x}$$

$$y'''' = 16c_3e^{2x} + 16c_4e^{-2x}$$

$$\therefore y^{(4)} - 4y'' = 16c_3e^{2x} + 16c_4e^{-2x} - 4(4c_3e^{2x} + 4c_4e^{-2x}) \\ = 0 \quad \checkmark$$

b. Use the operator (i.e. big 'D') method to solve for the general solution of the DE:

$$(D^4 - 4D^2)y = 0 \Rightarrow D^2(D^2 - 4)y = 0$$

$$\Rightarrow D^2(D-2)(D+2)y = 0$$

$$\Rightarrow D=0, 0, +2, -2$$

$$\therefore \boxed{y = c_1 + c_2x + c_3e^{2x} + c_4e^{-2x}} \quad \checkmark$$

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3. (15 pts) Solve the following:

a.  $\frac{dy}{dx} = \frac{1}{x} - \frac{3}{x}y, y(1) = 2$  (Hint: Separation of Variables)

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x}(1-3y) \Rightarrow \int \frac{dy}{1-3y} = \int \frac{1}{x} dx \Rightarrow -\frac{1}{3} \ln|1-3y| = \ln x + C \\ \Rightarrow e^{\ln|1-3y|} &= e^{\ln x + C} \Rightarrow \left(\frac{1}{1-3y}\right)^{1/3} = e^{\ln x} e^C = cx \\ \Rightarrow \frac{1}{1-3y} &= cx^3 \Rightarrow 1-3y = cx^{-3} \Rightarrow 3y = 1-cx^{-3} \\ \Rightarrow y &= \frac{1}{3} - cx^{-3} \Rightarrow y(1) = \frac{1}{3} - c = 2 \Rightarrow c = -\frac{5}{3} \\ \Rightarrow y &= \frac{1}{3} + \frac{5}{3}x^{-3}\end{aligned}$$

b. Using the solution from part a, calculate  $y(2)$ .

$$y(2) = \frac{1}{3} + \frac{5}{3}\left(\frac{1}{8}\right) = \frac{1}{3} - \frac{5}{4} = \frac{8}{24} + \frac{5}{24} = \boxed{\frac{13}{24} \approx .5417}$$

c. Using the initial condition  $y(1) = 1$ , use Euler's method to approximate  $y(2)$  with  $\Delta x = .5$ . Why is this answer different from part b?

$x$	$y$	$\frac{dy}{dx}$	$\Delta x$	$\Delta y$
1	1	-2	$\frac{1}{2}$	-1
1.5	0	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$
2	$\frac{1}{3}$			

Euler's Method is an approximation  
 $\Rightarrow$  also used wrong I.V.  $y(1) = 1$  vice  $y(1) = 2$

4. (10 pts) The half-life of a radioactive isotope is 15 years. When will 90% of the original mass of the isotope decay? Use the population model  $\frac{dm}{dt} = rm$  where  $m(0) = m_0$ .

$$\int \frac{dm}{m} = \int r dt \Rightarrow \ln m = rt + C \Rightarrow m = e^{rt+C} = e^{rt} e^C$$

$$\Rightarrow m = Ae^{rt}$$

$$\Rightarrow m(0) = A(1) = m_0 \Rightarrow m = m_0 e^{rt}$$

$$\Rightarrow m(15) = m_0 e^{15r} = \frac{1}{2}m_0 \Rightarrow r = \frac{1}{15} \ln\left(\frac{1}{2}\right) \approx -0.0462$$

$$\Rightarrow m = m_0 e^{-0.0462t}$$

$$\Rightarrow m(t) = m_0 e^{-0.0462t} = 0.1 m_0$$

$$\Rightarrow t = \frac{1}{0.0462} \ln(0.1) \Rightarrow t \approx 49.83 \text{ yrs}$$

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5. (10 pts) Solve the following 2<sup>nd</sup> order homogeneous linear differential equations. If we assume that the differential equation represents the equation of motion for a mass-spring system, characterize the damping (i.e. un-, over-, critically-, under damped).

a.  $\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 16y = 0$

$$(D^2 + 8D + 16)y = 0$$

$$\Rightarrow (D+4)^2 = 0 \Rightarrow D = -4, -4$$

$$\Rightarrow \boxed{y = C_1 e^{-4t} + C_2 t e^{-4t}} \quad \text{"repeated roots"}$$

"critically damped"

b.  $\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0$

$$D^2 + 7D + 10 = 0 \Rightarrow (D+2)(D+5) = 0$$

$$D = -2, -5 \Rightarrow \boxed{y = C_1 e^{-2t} + C_2 e^{-5t}}$$

"over damped"

6. (10 pts) Solve the following 2<sup>nd</sup> order homogeneous IVPs. If the equation represented the equation of motion for a mass-spring system, characterize the damping (i.e. un-, over-, critically-, under damped).

a.  $\frac{d^2y}{dt^2} + 9y = 0, y(0) = 1, y'(0) = 2.$

$$D^2 + 9 = 0$$

$$D = \pm 3i$$

$$y = C_1 \sin(3t) + C_2 \cos(3t)$$

$$y(0) = C_1 \sin(0) + C_2 \cos(0) = 1 \Rightarrow C_2 = 1$$

$$\Rightarrow y = C_1 \sin(3t) + \cos(3t)$$

$$\Rightarrow y' = 3C_1 \cos(3t) - 3 \sin(3t)$$

$$y' = 3C_1 = 2 \Rightarrow \boxed{C_1 = 2/3}$$

$$\boxed{y = \frac{2}{3} \sin(3t) + \cos(3t)}$$

"undamped"

b.  $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 25y = 0, y(0) = 1, y'(0) = 2.$

$$\frac{-6 \pm \sqrt{36 - 100}}{2} = \frac{-6 \pm \sqrt{-64}}{2} = \frac{-6 \pm 8i}{2} = -3 \pm 4i$$

$$y = C_1 e^{-3t} \sin(4t) + C_2 e^{-3t} \cos(4t)$$

$$y(0) = C_1(1)\cancel{\omega} + C_2(1)\cancel{1} = 1 \Rightarrow C_2 = 1$$

$$y = C_1 e^{-3t} \sin(4t) + e^{-3t} \cos(4t)$$

$$y' = -3C_1 e^{-3t} \sin(4t) + 4C_1 e^{-3t} \cos(4t) - 3e^{-3t} \cos(4t) - 4e^{-3t} \sin(4t)$$

$$y'(0) = -3C_1 \cancel{\omega} + 4(C_1 \cancel{1}) - 3\cancel{(1)}C_1 - 4\cancel{(1)}\cancel{\omega} = 2$$

$$\Rightarrow 4C_1 - 3 = 2 \Rightarrow C_1 = \frac{5}{4}$$

$$\therefore \boxed{y = \frac{5}{4} e^{-3t} \sin(4t) + e^{-3t} \cos(4t)}$$

"under damped"

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7. (10 pts) A 16 lb weight is attached to an undamped mass-spring system whose spring constant is 8 lbs/ft. Find the equation of motion if the weight is displaced 2 ft below equilibrium with a downward velocity of 1 ft/sec. Express your solution in the form  $y = A\sin(\omega t + \phi)$ . When does the weight 1<sup>st</sup> pass through equilibrium?

$$x=0 \quad k=8 \Rightarrow M = \frac{16}{32} = \frac{1}{2}$$

$$\therefore \frac{1}{2}x'' + 8x = 0 \Rightarrow x(0)=2, x'(0)=1$$

$$\Rightarrow x'' + 16x = 0 \Rightarrow (0^2 + 16)x = 0 \Rightarrow D = \pm 4i$$

$$x = C_1 \sin(4t) + C_2 \cos(4t)$$

$$x(0) = C_1(0) + C_2(1) = 2 \Rightarrow C_2 = 2$$

$$x = C_1 \sin(4t) + 2 \cos(4t) = 0$$

$$x' = 4C_1 \cos(4t) - 8 \sin(4t)$$

$$x'(0) = 4C_1(1) - 8(0) = 1 \Rightarrow C_1 = \frac{1}{4}$$

$$\therefore x = \frac{1}{4} \sin(4t) + 2 \cos(4t) \Rightarrow A = \sqrt{\left(\frac{1}{4}\right)^2 + 4} = \sqrt{\frac{1}{16} + 4} = \sqrt{\frac{65}{16}} = \frac{\sqrt{65}}{4} \approx 2.016$$

$$\therefore x \approx 2.016 \sin(4t + 1.446)$$

$$\phi \approx \tan^{-1}\left(\frac{2}{1/4}\right) = \tan^{-1}(8) \approx 1.446$$

$$4t + 1.446 = n\pi \quad \leftarrow \text{let } n=1 \text{ for 1st } t>0$$

$$\therefore t = \frac{\pi - 1.446}{4} \Rightarrow t \approx .4239 \text{ sec}$$