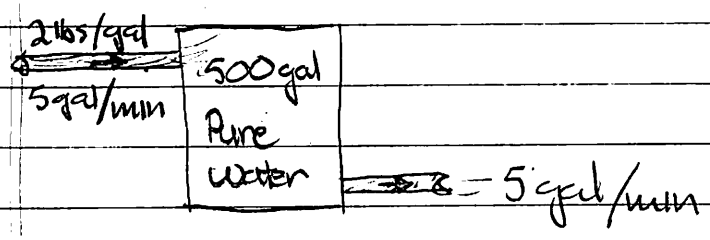
HW Solutions

Section 3.1 # 23, 25, 35
Mixing + Air Restance

23)



$A(t)$ = lbs of salt in tank

$A(0) = 0$ (pure water)

C = concentration of flow

f = flow rate

$$\frac{dA}{dt} = C_i f_i - C_o f_o$$

$C_i = 2 \text{ lbs/gal}$ $\quad C_o = A/500 \text{ lbs/gal}$

$f_i = 5 \text{ gal/min}$ $\quad f_o = 5 \text{ gal/m}$

"concentration = lbs salt divide by volume in tank"

$$\frac{dA}{dt} = 10 \frac{\text{lbs}}{\text{min}} - \frac{A}{100} \frac{\text{lbs}}{\text{m}} \Rightarrow \frac{dA}{dt} = 10 - .01A$$

integrating factors

$$\Rightarrow \frac{dA}{dt} + .01A = 10 \Rightarrow \mu(t) = e^{\int .01 dt} = e^{.01t}$$

$$\Rightarrow e^{.01t}\frac{dA}{dt} + .01e^{.01t}A = 10e^{.01t}$$

$$\Rightarrow \int \frac{d}{dt}\left[e^{.01t}A\right] = \int 10e^{.01t}\,dt$$

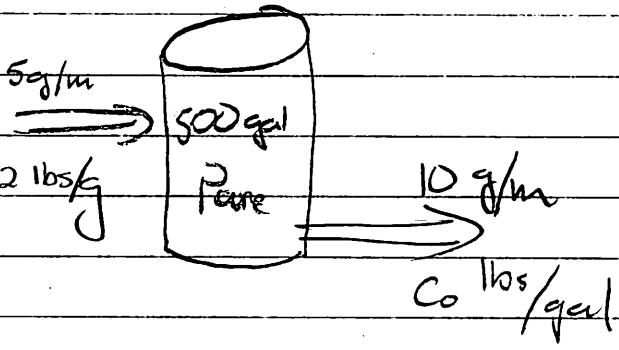
$$\Rightarrow e^{.01t}A = 1000e^{.01t} + C$$

$$\Rightarrow A = 1000 + Ce^{-.01t} \Rightarrow A(0) = 1000 + C = 0$$

$$C = 1000$$

$$\Rightarrow \boxed{A = 1000 - 1000e^{-.01t}} \Rightarrow \boxed{A = 1000(1 - e^{-.01t})}$$

25]



Note ⇒ Volume in the Tank =

$500 - 5t$

↑ "difference in flow rates"

Note: tank is empty in 100 minutes

$$\frac{dA}{dt} = 10\,\frac{lbs}{m} - \left(\frac{A\ lbs}{500-5t\ gal}\right)\left(10\,\frac{gal}{m}\right)$$

$$\frac{dA}{dt} = 10 - \frac{2A}{100-t} \Rightarrow \frac{dA}{dt} + \frac{2}{100-t}A = 10$$

Integrating Factor $\mu(t) = e^{\int \frac{2}{100-t}\,dt}$

$= e^{-2\ln(100-t)} = e^{\ln\left(\frac{1}{(100-t)^2}\right)}$

$= \frac{1}{(100-t)^2}$

$$\Rightarrow \frac{1}{(100-t)^2}\frac{dA}{dt} + \frac{2}{(100-t)^3}A = \frac{10}{(100-t)^2}$$

$$\Rightarrow \int \frac{d}{dt}\left[\frac{1}{(100-t)^2}A\right] = \int \frac{10}{(100-t)^2}\,dt$$

$$\frac{1}{(100-t)^2}A = \frac{10}{(100-t)} + C$$

$$\Rightarrow A = 10(100-t) + C(100-t)^2$$

$$A(0) = 10(100) + C(100)^2 = 0 \Rightarrow C = \frac{-10}{100} = -.1$$

$$\boxed{A = 10(100-t) - .1(100-t)^2}$$

$$A = 1000 - 10t - .1(1000 - 2t + t^2)$$

$$\Rightarrow A = 1000 - 10t - 100 + .2t - .1t^2$$

$$\Rightarrow \boxed{A = 900 - 9.8t - .1t^2}$$

35) $m\frac{dv}{dt} + kv = mg \Rightarrow \frac{dv}{dt} + \frac{k}{m}v = g$

$$\Rightarrow \mu(t) = e^{\int \frac{k}{m} dt} = e^{\frac{k}{m}t}$$

$$\Rightarrow e^{\frac{k}{m}t}\frac{dv}{dt} + \frac{k}{m}e^{\frac{k}{m}t}v = e^{\frac{k}{m}t}g$$

$$\Rightarrow \int \frac{d}{dt}\left[e^{\frac{k}{m}t}v\right] = \int e^{\frac{k}{m}t} g \, dt$$

$$\Rightarrow e^{\frac{k}{m}t}v = \frac{gm}{k}e^{\frac{k}{m}t} + C$$

$$\Rightarrow v = \frac{gm}{k} + Ce^{-\frac{k}{m}t}$$

a) $v(0) = \frac{gm}{k} + C = v_0 \Rightarrow C = v_0 - \frac{gm}{k}$

$$\Rightarrow \boxed{v = \frac{gm}{k} + \left(v_0 - \frac{gm}{k}\right)e^{-\frac{k}{m}t}}$$

b) as $t \to \infty \Rightarrow e^{-\frac{k}{m}t} \to 0 \Rightarrow \boxed{v_\infty = \frac{gm}{k}}$ ← terminal velocity

c) $v = \frac{ds}{dt} \Rightarrow ds = v\,dt \Rightarrow s = \int v\,dt$

$$\Rightarrow s = \int \left[\frac{gm}{k} + \left(v_0 - \frac{gm}{k}\right) e^{-\frac{k}{m}t} \right] dt$$

$$\Rightarrow s = \frac{gm}{k}t + \frac{-m}{k}\left(v_0 - \frac{gm}{k}\right) e^{-\frac{k}{m}t} + C$$

$$\Rightarrow s(0) = \frac{gm}{k}(0) - \frac{m}{k}\left(v_0 - \frac{gm}{k}\right) e^{0} + C = 0$$

$$\Rightarrow -\frac{m}{k}v_0 + \frac{gm^2}{k^2} + C = 0 \Rightarrow \boxed{C = \frac{m}{k}v_0 - \frac{gm^2}{k^2}}$$

$$\boxed{s = \frac{gm}{k}t - \frac{m}{k}\left(v_0 - \frac{gm}{k}\right) e^{-\frac{k}{m}t} + \left(\frac{m}{k}v_0 - \frac{gm^2}{k^2}\right)}$$